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Mixed correlation functions in the 2-matrix model, and the Bethe Ansatz

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Abstract: Using loop equation technics, we compute all mixed traces correlation functions of the 2-matrix model to large N leading order. The solution turns out to be a sort of Bethe Ansatz, i.e. all correlation functions can be decomposed on products of 2-point functions. We also find that, when the correlation functions are written collectively as a matrix, the loop equations are equivalent to commutation relations.

Keywords: Differential and Algebraic Geometry, Matrix Models, Bethe Ansatz.

Contents

1. Introduction

Formal random matrix models have been used for their interpretation as combinatorial generating functions for discretized surfaces [1 − 3]. The hermitean one-matrix model counts surfaces made of polygons of only one color, whereas the hermitean two-matrix model counts surfaces made of polygons of two colors. In that respect, the 2-matrix model is more appropriate for the purpose of studying surfaces with non-uniform boundary conditions. At the continuum limit, the 2-matrix model gives access to "boundary operators" in conformal field theory [4].

Generating functions for surfaces with boundaries are obtained as random matrix expectation values. The expectation value of a product of l traces is the generating function for surfaces with l boundaries, the total power of matrices in each trace being the length of the corresponding boundary. If each trace contains only one type of matrix (different traces may contain different types of matrices), the expectation value is the generating function counting surfaces with uniform boundary conditions. Those non-mixed expectation values have been computed for finite n since the work of [5, 6] and refined by [7].

Mixed correlation functions have been considered as a difficult problem for a long time and progress have been obtained only recently [8, 9]. Indeed, non-mixed expectation values can easily be written in terms of eigenvalues only (since the trace of a matrix is clearly related to its eigenvalues), whereas mixed correlation functions cannot ($TrM_1^k M_2^{k'}$ cannot be written in terms of eigenvalues of M_1 and M_2).

The large-N limit of the generating function of the bicolored disc (i.e. one boundary, two colors, i.e. $\langle TrM_1^k M_2^{k'}\rangle$) has been known since [10 − 12]. The large-N limit of the generating function of the 4-colored disc (i.e. one boundary, 4 colors, i.e. $\langle TrM_1^k M_2^{k'} M_1^{k''} M_2^{k'''}\rangle$) has been known since [13]. The all order expansion of correlation functions for the 1-matrix model has been obtained by a Feynman-graph representation in [14] and the generalization to non-mixed correlation functions of the 2-matrix model has been obtained in [15].

Recently, the method of integration over the unitary group of [9] has allowed to compute, for finite N, all mixed correlation functions of the 2-matrix model in terms of orthogonal polynomials.

The question of computing mixed correlation functions in the large-N limit is addressed in the present article.

The answer is (not so) surprisingly related to classical results in integrable statistical models, i.e. the Bethe Ansatz. It has been known for a long time that random matrix models are integrable in some sense (Toda, KP, KdV, isomonodromic systems,. . .), but the relationship with Yang-Baxter equations and Bethe Ansatz was rather indirect. The result presented in this article should give some new insight in that direction. We find that the k-point functions can be expressed as the product of 2-point function, which is the underlying idea of the Bethe Ansatz.

Outline of the article:

- section 1 is an introduction,

- in section 2, we set definitions of the model and correlation functions, and we write the relevant loop equations,
- in section 3, we introduce a Bethe Ansatz-like formula, and prove it in section 4,
- in section 5, we solve the problem under a matrix form,
- section 6 is dedicated to the special gaussian case.

2. The 2-matrix model, definitions and loop equations

2.1 Partition function

We are interested in the formal matrix integral:

$$Z := \int_{H_N^2} dM_1\, dM_2\, \mathrm{e}^{-N\,\mathrm{Tr}[V_1(M_1)+V_2(M_2)+M_1 M_2]} \tag{2.1}$$

where M_1 and M_2 are $N \times N$ hermitean matrices and dM_1 (resp. dM_2) is the product of Lebesgue measures of all independent real components of M_1 (resp. M_2). $V_1(x)$ and $V_2(y)$ are complex polynomials of degree d_1+1 and d_2+1, called "potentials". The formal matrix integral is defined as a formal power series in the coefficients of the potentials (see [3]), computed by the usual Feynman method: consider a local extremum of $\mathrm{e}^{-N\,\mathrm{Tr}[V_1(M_1)+V_2(M_2)+M_1 M_2]}$, and expand the non quadratic part as a power series and, for each term of the series, perform the gaussian integration with the quadratic part. This method does not care about the convergence of the integral, or of the series, it makes sense only order by order and it is in that sense that it can be interpreted as the generating function of discrete surfaces. All quantities in that model have a well defined $1/N^2$ expansion [16].

The extrema of $V_1(x)+V_2(y)+xy$ are such that:

$$V_1'(x) = -y\,, \qquad V_2'(y) = -x \tag{2.2}$$

there are $d_1 d_2$ solutions (indeed $V_2'(-V_1'(x)) = -x$), which we note $(\overline{x}_I, \overline{y}_I)$, $I = 1, \dots, d_1 d_2$. The extrema of $\mathrm{Tr}[V_1(M_1)+V_2(M_2)+M_1 M_2]$ can be chosen diagonal (up to a unitary transformation), with $\overline{x}_I$'s and $\overline{y}_I$'s on the diagonal:

$$\begin{aligned} M_1 &= \mathrm{diag}(\overbrace{\overline{x}_1,\dots,\overline{x}_1}^{n_1 \text{ times}}, \overbrace{\overline{x}_2,\dots,\overline{x}_2}^{n_2 \text{ times}}, \dots, \overbrace{\overline{x}_{d_1 d_2},\dots,\overline{x}_{d_1 d_2}}^{n_{d_1 d_2} \text{ times}}) \\ M_2 &= \mathrm{diag}(\overbrace{\overline{y}_1,\dots,\overline{y}_1}^{n_1 \text{ times}}, \overbrace{\overline{y}_2,\dots,\overline{y}_2}^{n_2 \text{ times}}, \dots, \overbrace{\overline{y}_{d_1 d_2},\dots,\overline{y}_{d_1 d_2}}^{n_{d_1 d_2} \text{ times}})\,. \end{aligned} \tag{2.3}$$

The extremum around which we perform the expansion is thus characterized by a set of filling fractions:

$$\epsilon_I = \frac{n_I}{N}\,, \qquad \sum_{I=1}^{d_1 d_2} \epsilon_I = 1\,. \tag{2.4}$$

JHEP08(2005)028

To summarize, let us say that the formal matrix integral is defined for given potentials and filling fractions.

The "one-cut" case is the one where one of the filling fractions is 1, and all the others vanish. This is the case where the Feynman expansion is performed in the vicinity of only one extremum.

2.2 Enumeration of discrete surfaces

It is well known that formal matrix integrals are generating functions for the enumeration of discrete surfaces [3, 2, 17, 18].

For instance, in the one-cut case (expansion near an extremum $\overline{x},\overline{y}$), one has:

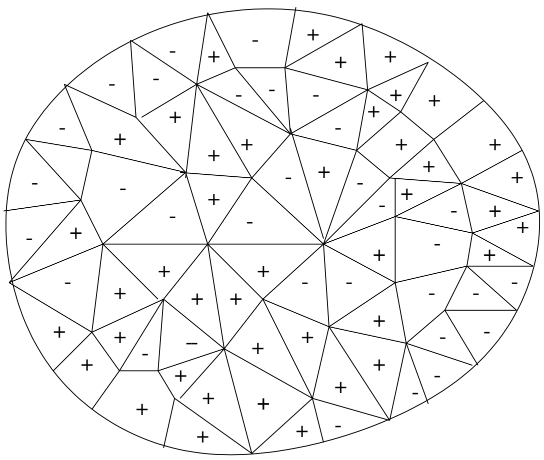
$$-\ln Z = \sum_G \frac{1}{\#\mathrm{Aut}(G)} N^{\chi(G)} \left(\frac{g_2}{\delta}\right)^{n_{--}(G)} \left(\frac{\tilde{g}_2}{\delta}\right)^{n_{++}(G)} \left(\frac{-1}{\delta}\right)^{n_{+-}(G)} \prod_{i=3}^{d_1+1} g_i^{n_i(G)} \prod_{i=3}^{d_2+1} \tilde{g}_i^{\tilde{n}_i(G)} \tag{2.5}$$

where the summation is over all finite connected closed discrete surfaces made of polygons of two signs (+ and -). For such a surface (or graph) G, $\chi(G)$ is its Euler characteristic, $n_i(G)$ is the number of i-gons carrying a + sign, $\tilde{n}_i(G)$ is the number of i-gons carrying a $-$ sign, $n_{++}(G)$ is the number of edges separating two + polygons, $n_{--}(G)$ is the number of edges separating two $-$ polygons and $n_{+-}(G)$ is the number of edges separating two polygons of different signs. $\#\mathrm{Aut}(G)$ is the number of automorphisms of G.

The g_i's, $\tilde{g}_i$'s and δ are defined as follows:

$$g_k := \left.\frac{\partial^k V_1(x)}{\partial x^k}\right|_{x=\overline{x}}, \qquad \tilde{g}_k := \left.\frac{\partial^k V_2(y)}{\partial x^k}\right|_{x=\overline{y}}, \qquad \delta := g_2\tilde{g}_2 - 1\,. \tag{2.6}$$

Example of a discrete surface:



(2.7)

In the multicut case, i.e. with arbitrary filling fractions, matrix integrals can still be interpreted in terms of "foams" of surfaces, and we refer the reader to the appendix of [19] or to [20] for more details.

2.3 Enumeration of discrete surfaces with boundaries

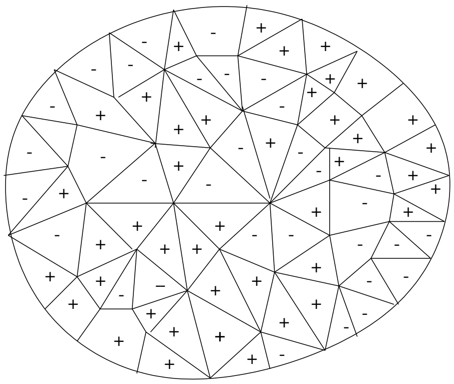
Similarly, given a sequence of signs $s_1, s_2, \dots, s_k$, $s_i \in 1, 2$, it is well known that the following quantity:

$$\left\langle \mathrm{Tr}\Big(\prod_{i=1}^{k} M_{s_i}\Big) \right\rangle \tag{2.8}$$

is the generating function of discrete surfaces with one boundary of length k, whose signs of polygons on the edges are given by the sequence $(s_1, \dots, s_k)$.

Example of a discrete surface with boundary $(++++++------+++++------)$:

$$\left\langle \mathrm{Tr}(M_1^6 M_2^5 M_1^5 M_2^6) \right\rangle = \sum_G \tag{2.9}$$



More generally, an expectation value of a product of n traces is the generating function for discrete surfaces with n boundaries.

In this article, we are interested only in one boundary and to leading order in N, i.e. surfaces with the topology of a disc.

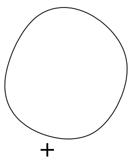
2.4 Master loop equation and algebraic curve

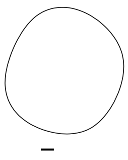
Let us define:

$$W(x) := \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{1}{x - M_1} \right\rangle, \qquad \tilde{W}(y) := \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{1}{y - M_2} \right\rangle \tag{2.10}$$

where the expectation values are formally computed as explained in the previous section, with the weight $\mathrm{e}^{-N\,\mathrm{Tr}[V_1(M_1)+V_2(M_2)+M_1M_2]}$. $W(x)$ (resp. $\tilde{W}(y)$) is defined as a formal power series in its large x (resp. large y) expansion, as well as in the expansion in the coefficients of the potentials. $W(x)$ (resp. $\tilde{W}(y)$) is a generating function for surfaces with one uniform boundary, i.e. with only sign $+$ (resp. sign $-$) polygons touching the boundary by an edge:

$$W(x) = \qquad , \qquad \tilde{W}(y) = \tag{2.11}$$





We also define the following formal series:

$$Y(x) := W(x) - V_1'(x)\,, \qquad X(y) := \tilde{W}(y) - V_2'(y)\,. \tag{2.12}$$

In addition, we define:

$$P(x,y) := \frac{1}{N}\left\langle \mathrm{Tr}\, \frac{V_1'(x)-V_1'(M_1)}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right\rangle \tag{2.13}$$

$$U(x,y) := \frac{1}{N}\left\langle \mathrm{Tr}\, \frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right\rangle + x + V_2'(y) \tag{2.14}$$

$$\begin{aligned} U(x,y;x') :=& \left\langle \mathrm{Tr}\, \frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\,\mathrm{Tr}\,\frac{1}{x'-M_1}\right\rangle - \\ & -N^2 W(x')(U(x,y)-x-V_2'(y)) \end{aligned} \tag{2.15}$$

$$E(x,y) := (V_1'(x)+y)(V_2'(y)+x) + P(x,y) - 1\,. \tag{2.16}$$

Notice that $U(x,y)$ and $U(x,y;x')$ are polynomials of y (with degree at most d_2-1), $P(x,y)$ is a polynomial of both variables of degree (d_1-1, d_2-1) and $E(x,y)$ is a polynomial of both x and y of degree (d_1+1, d_2+1).

It has been obtained in many articles [21, 22, 11, 12], that:

$$E(x,Y(x)) = \frac{1}{N^2}U(x,Y(x),x)\,. \tag{2.17}$$

To large-N leading order that equation reduces to an algebraic equation for $Y(x)$, called the "Master loop equation" [22]:

$$E(x,Y(x)) = 0 \tag{2.18}$$

(similarly, one also has $E(X(y),y)=0$, which implies $Y\circ X = \mathrm{Id}$, known as Matytsin's equation [23]). The coefficients of $E(x,y)$, i.e. of $P(x,y)$, are entirely determined by the conditions $\oint_{\mathcal{A}_i} y dx = 2i\pi\epsilon_i$ for a choice of irreducible cycles on the algebraic curve.

The properties of that algebraic equation have been studied in many works [21, 24]. Here we assume that it is known.

2.5 Correlation functions, definitions

We define:

$$\overline{W}_k(x_1,y_1,x_2,\dots,x_k,y_k) := \frac{1}{N}\left\langle \mathrm{Tr}\prod_{j=1}^{k}\frac{1}{x_j-M_1}\frac{1}{y_j-M_2}\right\rangle \tag{2.19}$$

$$\begin{aligned} \overline{U}_k(x_1,y_1,x_2,\dots,x_k,y_k) :=& \mathop{\mathrm{Pol}}_{y_k} V_2'(y_k)\,\overline{W}_k(x_1,y_1,x_2,\dots,x_k,y_k) \\ =& \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{1}{x_1-M_1}\frac{1}{y_1-M_2}\cdots\frac{1}{x_k-M_1}\frac{V_2'(y_k)-V_2'(M_2)}{y_k-M_2}\right\rangle \end{aligned} \tag{2.20}$$

$$\begin{aligned} \overline{P}_k(x_1,y_1,x_2,\dots,x_k,y_k) :=& \mathop{\mathrm{Pol}}_{x_1}\mathop{\mathrm{Pol}}_{y_k} V_1'(x_1)\,V_2'(y_k)\,\overline{W}_k(x_1,y_1,x_2,\dots,x_k,y_k) \\ =& \frac{1}{N}\Bigg\langle \mathrm{Tr}\,\frac{V_1'(x_1)-V_1'(M_1)}{x_1-M_1}\,\frac{1}{y_1-M_2}\times\cdots \\ & \cdots\times\frac{1}{x_k-M_1}\,\frac{V_2'(y_k)-V_2'(M_2)}{y_k-M_2}\Bigg\rangle \end{aligned} \tag{2.21}$$

$$A_k(x_1,y_1,x_2,\dots,x_k) := \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{1}{x_1-M_1}\frac{1}{y_1-M_2}\cdots\frac{1}{x_k-M_1}V_2'(M_2)\right\rangle \tag{2.22}$$

where $\mathrm{Pol}_x f(x)$ denotes the polynomial part at infinity of $f(x)$ (i.e. the positive part in the Laurent series for x near infinity).

The functions $\overline{W}_k$ are generating functions for discrete discs with all possible boundary conditions. One can recover any generating function of type eq. (2.8) by expanding into powers of the x_i's and y_i's.

For convenience, we prefer to consider the following functions:

$$W_k(x_1, y_1, x_2, \ldots, x_k, y_k) := \overline{W}_k(x_1, y_1, x_2, \ldots, x_k, y_k) + \delta_{k,1} \tag{2.23}$$

$$U_k(x_1, y_1, x_2, \ldots, x_k, y_k) := \overline{U}_k(x_1, y_1, x_2, \ldots, x_k, y_k) + \delta_{k,1}(V_2'(y_k) + x_k) \tag{2.24}$$

and for $k > 1$:

$$P_k(x_1, y_1, x_2, \ldots, x_k, y_k) := \overline{P}_k(x_1, y_1, x_2, \ldots, x_k, y_k) + W_{k-1}(x_2, \ldots, x_k, y_1) \tag{2.25}$$

For the smallest values of k, those expectation values can be found in the literature to large-N leading order:

- it was found in [11, 12, 21]:

$$W_1(x, y) = \frac{E(x, y)}{(x - X(y))(y - Y(x))}, \qquad U_1(x, y) = \frac{E(x, y)}{(y - Y(x))} \tag{2.26}$$

- it was found in the appendix C of [13] (there is a change of sign, because the action in [13] was $\mathrm{e}^{-N\,\mathrm{tr}(V_1(M_1)+V_2(M_2)-M_1M_2)}$):

$$W_2(x_1, y_1, x_2, y_2) = \frac{W_1(x_1, y_1)W_1(x_2, y_2) - W_1(x_1, y_2)W_1(x_2, y_1)}{(x_1 - x_2)(y_1 - y_2)} \tag{2.27}$$

- for finite N, it was found in [8], and with notations explained in [8]:

$$W_1(x, y) = \det\left(1_N + \Pi_{N-1}\frac{1}{x - Q}\frac{1}{y - P^t}\Pi_{N-1}\right) \tag{2.28}$$

- for finite N, it was found in [9] how to compute any mixed correlation function in terms of determinants involving biorthogonal polynomials, with a formula very similar to eq. (2.28).

Here, we shall find a formula for all W_k's in the large-N limit.

2.6 Loop equations

Loop equations are nothing but Schwinger-Dyson equations. They are obtained by writing that an integral is invariant under a change of variable, or alternatively by writing that the integral of a total derivative vanishes.

The loop equation method is well known and explained in many works [21, 22]. Here, we write for each change of variable the corresponding loop equation (we use a presentation similar to that of [21]).

In all what follows we consider $k > 1$.

JHEP08(2005)028

- the change of variable: $\delta M_2 = \frac{1}{x_1-M_1}\frac{1}{y_1-M_2}\,\cdots\,\frac{1}{x_k-M_1}$ implies:

$$\begin{aligned}
A_k(x_1,\dots,x_k) &= \sum_{j=1}^{k-1}\overline{W}_j(x_1,\dots,y_j)\,\overline{W}_{k-j}(x_k,y_j,\dots,y_{k-1})+\\
&+\frac{x_1\overline{W}_{k-1}(x_1,y_1,\dots,y_{k-1})-x_k\overline{W}_{k-1}(x_k,y_1,\dots,y_{k-1})}{x_1-x_k}\\
&=\sum_{j=1}^{k-1}W_j(x_1,\dots,y_j)\,W_{k-j}(x_k,y_j,\dots,y_{k-1})-\\
&-W_{k-1}(x_k,y_1,\dots,y_{k-1})+\\
&+x_k\,\frac{W_{k-1}(x_1,y_1,\dots,y_{k-1})-W_{k-1}(x_k,y_1,\dots,y_{k-1})}{x_1-x_k}
\end{aligned}\tag{2.29}$$

- the change of variable: $\delta M_1 = \frac{1}{x_1-M_1}\frac{1}{y_1-M_2}\,\cdots\,\frac{1}{x_k-M_1}\frac{V_2'(y_k)-V_2'(M_2)}{y_k-M_2}$ implies:

$$\begin{aligned}
&(Y(x_1)-y_k)\,\overline{U}_k(x_1,\dots,y_k)=\\
&\quad=\sum_{j=2}^{k}\frac{W_{j-1}(x_1,y_1,\dots,y_{j-1})-W_{j-1}(x_j,y_1,\dots,y_{j-1})}{x_1-x_j}\,\overline{U}_{k-j+1}(x_j,y_j,\dots,x_k,y_k)+\\
&\quad+V_2'(y_k)\frac{W_{k-1}(x_1,y_1,\dots,y_{k-1})-W_{k-1}(x_k,y_1,x_2,\dots,y_{k-1})}{x_1-x_k}+\\
&\quad+A_k(x_1,\dots,x_k)-\overline{P}_k(x_1,y_1,x_2,\dots,x_k,y_k)\\
&\quad=\sum_{j=2}^{k}\frac{W_{j-1}(x_1,y_1,\dots,y_{j-1})-W_{j-1}(x_j,y_1,\dots,y_{j-1})}{x_1-x_j}\,U_{k-j+1}(x_j,y_j,\dots,x_k,y_k)+\\
&\quad+\sum_{j=1}^{k-1}W_j(x_1,\dots,y_j)\,W_{k-j}(x_k,y_j,\dots,y_{k-1})-P_k(x_1,y_1,x_2,\dots,x_k,y_k)
\end{aligned}\tag{2.30}$$

where we have used the loop equation eq. (2.29) for $A_k(x_1,\dots,x_k)$.

- the change of variable: $\delta M_2 = \frac{1}{x_1-M_1}\frac{1}{y_1-M_2}\,\cdots\,\frac{1}{x_k-M_1}\frac{1}{y_k-M_2}$ implies:

$$\begin{aligned}
&(X(y_k)-x_1)\,\overline{W}_k(x_1,y_1,x_2,\dots,x_k,y_k)=\\
&\quad=\sum_{j=1}^{k-1}\frac{W_{k-j}(x_{j+1},\dots,y_k)-W_{k-j}(x_{j+1},\dots,x_k,y_j)}{y_k-y_j}\times\\
&\quad\times W_j(x_1,\dots,y_j)-U_k(x_1,\dots,y_k)\,.
\end{aligned}\tag{2.31}$$

2.7 Recursive determination of the correlation functions

Theorem 1. *The system of equations eq. (2.30) and eq. (2.31) for all k has a unique solution.*

In other words, if we can find some functions W_k, U_k, P_k which obey eq. (2.30) and eq. (2.31) for all k, then they are the correlation functions we are seeking.
Proof. W_1, U_1 and P_1 have already been computed in the literature.

JHEP08(2005)028

Assume that we have computed W_j, U_j, P_j for all $j < k$. Let us show that eq. (2.30) and eq. (2.31) determine uniquely W_k, U_k and P_k.

Let $X^{(\alpha)}(y_k)$, $\alpha = 0, \ldots, d_1$ be the $d_1 + 1$ solutions for x of $E(x, y_k) = 0$. For every $\alpha = 0, \ldots, d_1$ one has:

$$Y(X^{(\alpha)}(y_k)) = y_k \tag{2.32}$$

At $x_1 = X^{(\alpha)}(y_k)$, eq. (2.30) reads:

$$\begin{aligned} & P_k(X^{(\alpha)}(y_k), y_1, x_2, \ldots, x_k, y_k) = \\ & \quad = \sum_{j=2}^{k} \frac{W_{j-1}(X^{(\alpha)}(y_k), y_1, \ldots, y_{j-1}) - W_{j-1}(x_j, y_1, x_2, \ldots, y_{j-1})}{X^{(\alpha)}(y_k) - x_j} \times \\ & \qquad \times U_{k-j+1}(x_j, y_j, \ldots, x_k, y_k) + \sum_{j=1}^{k-1} W_j(X^{(\alpha)}(y_k), \ldots, y_j)\, W_{k-j}(x_k, y_j, \ldots, y_{k-1}) \end{aligned} \tag{2.33}$$

where all the quantities in the r.h.s. are known from the recursion hypothesis. We thus know the value of P_k for $d_1 + 1$ values of x_1. Since P_k is a polynomial in x_1 of degree at most $d_1 - 1$, we can determine P_k by the interpolation formula:

$$(x_1 - X(y_k)) \frac{P_k(x_1, \ldots, y_k)}{E(x_1, y_k)} = \sum_{\alpha=1}^{d_2} \frac{(X^{(\alpha)}(y_k) - X(y_k))\, P_k(X^{(\alpha)}(y_k), \ldots, y_k)}{(x_1 - X^{(\alpha)}(y_k))\, E_x(X^{(\alpha)}(y_k), y_k)} \tag{2.34}$$

where $X_k = X(y_k)$ denotes $X^{(0)}(y_k)$. Once P_k is known, eq. (2.30) allows to compute U_k, and eq. (2.31) allows to compute W_k.

3. A Bethe Ansatz-like formula for correlation functions

Thus, the loop equations determine W_k uniquely, i.e., if we can find W_k, $U_k = \mathrm{Pol}_{y_k} \times V_2'(y_k) W_k$ and $P_k = W_{k-1} + \mathrm{Pol}_{x_1} V_1'(x_1) U_k$ which satisfy eq. (2.31),eq. (2.30), it means that we have the right solution. We can thus make an Ansatz for W_k, and check that it satisfies the loop equations above.

Our Ansatz is similar to the Bethe Ansatz [25]:

$$W_k(x_1, y_1, \ldots, x_k, y_k) = \sum_{\sigma \in \overline{\Sigma}_k} C_\sigma^{(k)}(x_1, y_1, \ldots, x_k, y_k) \prod_{i=1}^{k} W_1(x_i, y_{\sigma(i)}) \tag{3.1}$$

where the coefficients C_σ are rational fractions of the x_i's and y_i's, with at most simple poles at coinciding points, and independent of the potentials.

We call eq. (3.1) a Bethe Ansatz-like formula, because it is very similar to the solution initially found by Bethe for the 1-dimensional spin chain, and then for the δ-interacting bosons.

If we assume that eq. (3.1) satisfies eq. (2.31), we can in particular take the residue of eq. (2.31) at $y_k \to Y(x_l)$ for some l. That implies the following relationship among the coefficients $C_\sigma^{(k)}$'s:

$$(x_{\sigma^{-1}(k)} - x_1)\, C_\sigma^{(k)}(x_1, y_1, \dots, x_k, y_k) = \\ = \sum_{j=1}^{k-1} \sum_{\tau \in \Sigma(1,\dots,j)} \sum_{\rho \in \Sigma(j+1,\dots,k)} \delta_{\sigma,\tau\rho} \, \frac{C_\tau^{(j)}(x_1,\dots,y_j) C_\rho^{(k-j)}(x_{j+1},\dots,y_k)}{y_k - y_j} \,. \tag{3.2}$$

Beside, since W_k is the expectation value of a trace, the $C_\sigma^{(k)}$'s must be cyclically invariant:

$$C_\sigma^{(k)}(x_1, y_1, x_2, y_2, \dots, x_k, y_k) = C_\sigma^{(k)}(x_2, y_2, \dots, x_k, y_k, x_1, y_1) \tag{3.3}$$

and, since W_k should have no poles at coinciding points $y_k = y_j$ one should have:

$$\operatorname*{Res}_{y'_k \to y_j} C_\sigma^{(k)}(x_1, y_1, x_2, y_2, \dots, x_k, y'_k) dy'_k = \operatorname*{Res}_{y'_k \to y_j} C_{(k,j)\circ\sigma}^{(k)}(x_1, y_1, x_2, y_2, \dots, x_k, y'_k) dy'_k \,. \tag{3.4}$$

With $C^{(1)} = 1$, it is clear that the set of equations eq. (3.2), eq. (3.3), eq. (3.4), have at most a unique solution. We prove in the next section that the solution exists, and thus, eq. (3.2), eq. (3.3), eq. (3.4) determine $C_\sigma^{(k)}$ uniquely. The $C_\sigma^{(k)}$'s are explicitly computed in section 4.

Then in section 4.7, we prove that:

Theorem 2. *If the $C_\sigma^{(k)}$'s are rational functions defined by eq. (3.2), eq. (3.3), eq. (3.4), then the functions $\widehat{W}_k$'s defined by the r.h.s. of eq. (3.1), the functions $\widehat{U}_k(x_1, \dots, y_k) := \mathrm{Pol}(V_2'(y_k))\widehat{W}_k(x_1, \dots, y_k)$, and the functions $\widehat{P}_k(x_1, \dots, y_k) := \mathrm{Pol}(V_1'(x_1))\widehat{U}_k(x_1, \dots, y_k) + \widehat{W}_{k-1}(x_2, \dots, x_k, y_1)$, satisfy eq. (2.31) and eq. (2.30).*

As a corollary, using theorem 1, we have:

Theorem 3.

$$\boxed{W_k(x_1, y_1, \dots, x_k, y_k) = \sum_{\sigma \in \overline{\Sigma}_k} C_\sigma^{(k)}(x_1, y_1, \dots, x_k, y_k) \prod_{i=1}^{k} W_1(x_i, y_{\sigma(i)}) \,.} \tag{3.5}$$

The derivation of theorem 2 is quite technical and is presented in section 4.7.

4. Amplitudes of permutations

In this section, we compute the amplitudes $C_\sigma^{(k)}$ explicitly.

Eq. (3.2), eq. (3.3), eq. (3.4) and initial condition $C^{(1)}(x_1, y_1) = 1$ clearly define at most a unique function $C_\sigma^{(k)}(x_1, \dots, y_k)$. In this section, we build the solution explicitly, and then, we prove that the function we have constructed indeed satisfies eq. (3.2), eq. (3.3), eq. (3.4).

Below we prove that eq. (3.2) implies that $C_\sigma^{(k)}$ vanishes for non planar permutations (see Definition 4.1 below) and, for planar permutations, $C_\sigma^{(k)}$ is the product of $C_{\mathrm{Id}}^{(k)}$ corresponding to faces. We are thus led to introduce the definitions in the next section.

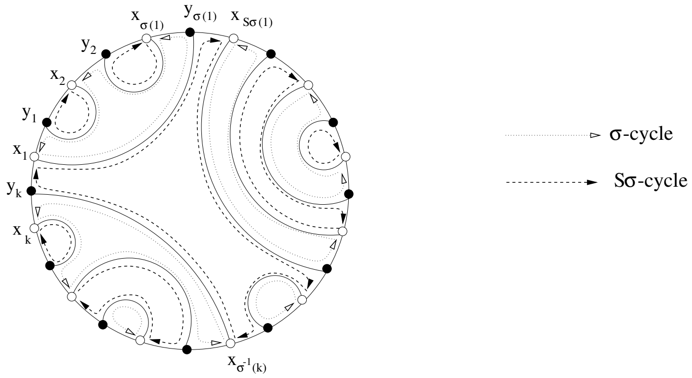


Figure 1: Example of a planar permutation and its faces.

4.1 Some definitions: planar permutations

Let S be the shift permutation:

$$S := \mathrm{shift} = (1, 2, \ldots, k-1, k)\,, \qquad \text{i.e.} \qquad S(i) = i + 1\,. \tag{4.1}$$

Definition 1. *A permutation $\sigma \in \Sigma_k$ is called planar if*

$$n_{\text{cycles}}(\sigma) + n_{\text{cycles}}(S\sigma) = k + 1 \tag{4.2}$$

where $n_{\text{cycles}}(\sigma)$ is the number of irreducible cycles composing the permutation σ.

Let $\overline{\Sigma}_k \subset \Sigma_k$ be the set of planar permutations of rank k.

Eq. (4.2) is equivalent to saying that if one draws the points $x_1, y_1, \ldots, x_k, y_k$ on a circle, and draws a line between each pair $(x_j, y_{\sigma(j)})$, the lines don't intersect. The cycles of σ and the cycles of $S\sigma$ correspond to the faces (i.e. the connected components) of that partition of the disc.

Each planar permutation can also be represented as an arch system, and thus, the number of possible planar permutations is given by the Catalan number $\mathrm{Cat}\,(k)$:

$$\mathrm{Card}(\overline{\Sigma}_k) = \mathrm{Cat}\,(k) = \frac{2k!}{k!\,(k+1)!} \tag{4.3}$$

4.2 Face amplitudes

Definition 2. *For any $k \geq 1$, we define a rational function of $x_1, \ldots, y_k$:*

$$F^{(k)}(x_1, y_1, x_2, \ldots, x_k, y_k) \tag{4.4}$$

by the recursion formula:

$$F^{(1)}(x_1, y_1) := 1 \tag{4.5}$$

$$F^{(k)}(x_1, y_1, \ldots, x_k, y_k) := \sum_{j=1}^{k-1} \frac{F^{(j)}(x_1, y_1, \ldots, x_j, y_j) F^{(k-j)}(x_{j+1}, y_{j+1}, \ldots, x_k, y_k)}{(x_k - x_1)(y_k - y_j)}\,.$$

Lemma 1. *For any $k \geq 1$:*

$$F^{(k)}(x_1, y_1, \ldots, x_k, y_k) = O(y_k^{1-k}) \tag{4.6}$$

when $y_k \to \infty$.

Proof. Let us prove it by induction on k. It is true for k=1. Let k be larger or equal to 2 and assume that this is true for all $F^{(j)}$ with $j < k$. Then eq. (4.5) straightforwardly gives the same behaviour for $F^{(k)}$.

Lemma 2. *$F^{(k)}$ has cyclic invariance, i.e.*

$$F^{(k)}(x_2, y_2, \ldots, x_k, y_k, x_1, y_1) = F^{(k)}(x_1, y_1, \ldots, x_k, y_k) \tag{4.7}$$

Proof. We prove it by recursion. It is clearly true for $k = 1$ and $k = 2$ since $F^{(2)} \times (x_1, y_1, x_2, y_2) = \frac{1}{(x_2-x_1)(y_2-y_1)}$. For $k \geq 3$, assume that it is true for all $F^{(j)}$ with $j < k$. One has:

$$\begin{aligned}
&F^{(k)}(x_2, y_2, \ldots, x_k, y_k, x_1, y_1) = \\
&\quad = \sum_{j=2}^{k} \frac{F^{(j-1)}(x_2, \ldots, x_j, y_j) F^{(k+1-j)}(x_{j+1}, y_{j+1}, \ldots, x_k, y_k, x_1, y_1)}{(x_1 - x_2)(y_1 - y_j)} \\
&\quad = \sum_{j=2}^{k} \frac{F^{(j-1)}(x_2, \ldots, x_j, y_j) F^{(k+1-j)}(x_1, y_1, x_{j+1}, y_{j+1}, \ldots, x_k, y_k)}{(x_1 - x_2)(y_1 - y_j)} \\
&\quad = \sum_{j=2}^{k} \frac{F^{(j-1)}(x_2, \ldots, x_j, y_j)}{(x_1 - x_2)(y_1 - y_j)} \times \\
&\qquad \times \sum_{l=j+1}^{k-1} \frac{F^{(l+1-j)}(x_1, y_1, x_{j+1}, y_{j+1}, \ldots, x_l, y_l) F^{(k-l)}(x_{l+1}, y_{l+1}, \ldots, x_k, y_k)}{(x_k - x_1)(y_k - y_l)} + \\
&\qquad + \sum_{j=2}^{k} \frac{F^{(j-1)}(x_2, \ldots, x_j, y_j)}{(x_1 - x_2)(y_1 - y_j)} \frac{F^{(1)}(x_1, y_1) F^{(k-j)}(x_{j+1}, y_{j+1}, \ldots, x_k, y_k)}{(x_k - x_1)(y_k - y_1)} \\
&\quad = \sum_{l=3}^{k-1} \frac{F^{(k-l)}(x_{l+1}, y_{l+1}, \ldots, x_k, y_k)}{(x_k - x_1)(y_k - y_l)} \times \\
&\qquad \times \sum_{j=2}^{l} \frac{F^{(j-1)}(x_2, \ldots, x_j, y_j) F^{(l+1-j)}(x_{j+1}, y_{j+1}, \ldots, x_l, y_l, x_1, y_1)}{(x_1 - x_2)(y_1 - y_j)} \\
&\quad = \sum_{l=3}^{k-1} \frac{F^{(k-l)}(x_{l+1}, y_{l+1}, \ldots, x_k, y_k) F^{(l)}(x_2, \ldots, x_l, y_l, x_1, y_1)}{(x_k - x_1)(y_k - y_l)} \\
&\quad = \sum_{l=3}^{k-1} \frac{F^{(l)}(x_1, y_1, x_2, \ldots, x_l, y_l) F^{(k-l)}(x_{l+1}, y_{l+1}, \ldots, x_k, y_k)}{(x_k - x_1)(y_k - y_l)} \\
&\quad = F^{(k)}(x_1, y_1, \ldots, x_k, y_k)\,.
\end{aligned} \tag{4.8}$$

Lemma 3. *For $k \geq 2$, $F^{(k)}$ has simple poles in y_k:*

$$F^{(k)}(x_1, y_1, \ldots, x_k, y_k) = \sum_{l=1}^{k-1} \frac{1}{y_k - y_l} \operatorname*{Res}_{y'_k \to y_l} F^{(k)}(x_1, y_1, \ldots, x_k, y'_k) dy'_k \,. \tag{4.9}$$

Proof. It is clearly true for $k = 2$. We prove it by induction on k. Assume that it is true up to $k-1$. Using the recursion hypothesis, one can see that each term in the r.h.s. of eq. (4.5) has at most a simple pole at $y_k = y_l$ and one can write eq. (4.9) with the use of Lemma 1. Thus the recursion hypothesis is true for k.

4.3 The amplitudes C_σ

Definition 3. *Then, for any $k \geq 1$, and for any permutation $\sigma \in \Sigma_k$, we define $C_\sigma^{(k)} \times (x_1, y_1, x_2, \ldots, x_k, y_k)$ a rational function of $x_1, \ldots, y_k$, by:*

- $C_\sigma^{(k)}(x_1, y_1, x_2, \ldots, x_k, y_k) := 0$ *if σ is not planar, and*
- *if σ is planar, we decompose σ and $S\sigma$ into their product of cycles:*

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_l \,, \qquad S\sigma = \tilde{\sigma}_1 \tilde{\sigma}_2 \ldots \tilde{\sigma}_{\tilde{l}} \tag{4.10}$$

such that:

$$\sigma_j = (i_{j,1}, i_{j,2}, \ldots, i_{j,l_j}) \,, \qquad \sigma(i_{j,m}) = i_{j,m+1} \tag{4.11}$$

$$\tilde{\sigma}_j = (\tilde{i}_{j,1}, \tilde{i}_{j,2}, \ldots, \tilde{i}_{j,\tilde{l}_j}) \,, \qquad \sigma(\tilde{i}_{j,m}) = \tilde{i}_{j,m+1} - 1 \tag{4.12}$$

$$\begin{aligned} C_\sigma^{(k)}(x_1, y_1, x_2, \ldots, x_k, y_k) := & \prod_{j=1}^{l} F^{(l_j)}(x_{i_{j,1}}, y_{i_{j,2}}, x_{i_{j,2}}, y_{i_{j,3}}, \ldots, x_{i_{j,l_j}}, y_{i_{j,1}}) \times \\ & \times \prod_{j=1}^{\tilde{l}} F^{(\tilde{l}_j)}(x_{\tilde{i}_{j,1}}, y_{\tilde{i}_{j,2}-1}, x_{\tilde{i}_{j,2}}, \ldots, y_{\tilde{i}_{j,\tilde{l}_j}-1}, x_{\tilde{i}_{j,\tilde{l}_j}}, y_{\tilde{i}_{j,1}-1}) \end{aligned} \tag{4.13}$$

i.e. $C_\sigma^{(k)}$ is the product of F's of each connected component of the disc partitioned by σ.

4.4 Examples

In particular with $\sigma = Id$, we have:

$$C_{Id}^{(k)}(x_1, y_1, x_2, \ldots, x_k, y_k) = F^{(k)}(x_1, y_1, x_2, y_2, \ldots, x_k, y_k) \tag{4.14}$$

and with $\sigma = S^{-1}$:

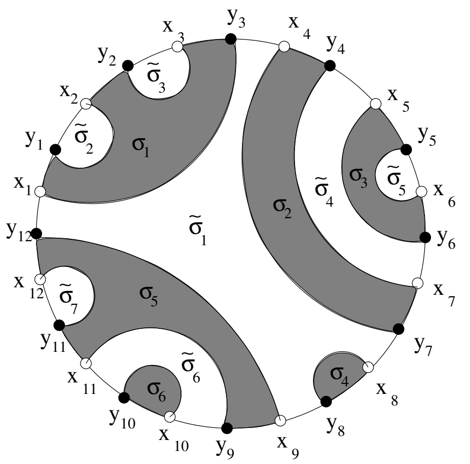
$$C_{S^{-1}}^{(k)}(x_1, y_1, x_2, \ldots, x_k, y_k) = F^{(k)}(x_k, y_{k-1}, x_{k-1}, \ldots, y_1, x_1, y_k) \,. \tag{4.15}$$

JHEP08(2005)028

An example with $k = 12$. Let us consider an example of $\sigma \in \overline{\Sigma}_{12}$ defined as follow: $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 2$, $\sigma(4) = 7$, $\sigma(5) = 6$, $\sigma(6) = 5$, $\sigma(7) = 4$, $\sigma(8) = 8$, $\sigma(9) = 11$, $\sigma(10) = 10$, $\sigma(11) = 12$ and $\sigma(12) = 9$:

$$\begin{aligned} \sigma &= (1,3,2)(4,7)(5,6)(8)(9,12,11)(10) \\ S\sigma &= (1,4,8,9)(2)(3)(5,7)(6)(10,11)(12)\,. \end{aligned} \tag{4.16}$$

The corresponding arch system is:



(4.17)

where dark faces (resp. white faces) correspond to the cycles of σ (resp. $S\sigma$). For that permutation $C_\sigma^{(12)}$ is worth:

$$\begin{aligned} &F^{(3)}(x_1,y_3,x_3,y_2,x_2,y_1)F^{(2)}(x_5,y_6,x_6,y_5)F^{(3)}(x_9,y_{12},x_{12},y_{11},x_{11},y_9)\times \\ &\times F^{(2)}(x_4,y_7,x_7,y_4)F^{(4)}(x_1,y_3,x_4,y_7,x_8,y_8,x_9,y_{12})F^{(2)}(x_5,y_6,x_7,y_4)\times \\ &\times F^{(2)}(x_{10},y_{10},x_{11},y_9) \end{aligned} \tag{4.18}$$

Example $k \leq 3$.

$$C^{(1)} = 1 \tag{4.19}$$

$$C^{(2)}_{Id} = \frac{1}{(x_2-x_1)(y_2-y_1)} \tag{4.20}$$

$$C^{(2)}_{(12)} = \frac{1}{(x_2-x_1)(y_1-y_2)} \tag{4.21}$$

$$C^{(3)}_{Id} = \frac{1}{(x_1-x_3)}\left(\frac{1}{y_3-y_1}\frac{1}{(x_2-x_3)(y_3-y_2)}+\frac{1}{y_3-y_2}\frac{1}{(x_1-x_2)(y_2-y_1)}\right) \tag{4.22}$$

$$C^{(3)}_{(1)(23)} = \frac{1}{(x_1-x_2)}\frac{1}{y_3-y_1}\frac{1}{(x_2-x_3)(y_2-y_3)} \tag{4.23}$$

$$C^{(3)}_{(12)(3)} = \frac{1}{(x_1-x_3)}\frac{1}{y_3-y_2}\frac{1}{(x_1-x_2)(y_1-y_2)} \tag{4.24}$$

$$\begin{aligned} C^{(3)}_{(123)} &= \frac{1}{(x_1-x_2)(x_2-x_3)(y_1-y_2)}\frac{1}{y_3-y_1}+ \\ &+\frac{1}{(x_1-x_3)(x_1-x_2)(y_2-y_1)}\frac{1}{y_3-y_2} \end{aligned} \tag{4.25}$$

$$C^{(3)}_{(13)(2)} = -\frac{1}{y_3-y_1}\frac{1}{(x_1-x_3)(x_2-x_3)(y_1-y_2)} \tag{4.26}$$

$$C^{(3)}_{(132)} = 0\,. \tag{4.27}$$

Example: rainbows. The rainbow is the permutation

$$\sigma(j) = k+1-j \tag{4.28}$$

if k is even:

$$C^{(k)}_\sigma = \frac{1}{\prod_{i=1}^{k/2}(x_{k+1-i}-x_i)(y_i-y_{k+1-i})\prod_{i=1}^{k/2-1}(x_{k+1-i}-x_{i+1})(y_i-y_{k-i})} \tag{4.29}$$

if k is odd:

$$C^{(k)}_\sigma = \frac{1}{\prod_{i=1}^{(k-1)/2}(x_{k+1-i}-x_i)(x_{k+1-i}-x_{i+1})(y_i-y_{k+1-i})(y_i-y_{k-i})}\,. \tag{4.30}$$

4.5 Properties of C_σ

Lemma 4. *The $C^{(k)}_\sigma$'s are cyclically invariant:*

$$C^{(k)}_\sigma(x_1,y_1,x_2,y_2,\dots,x_k,y_k) = C^{(k)}_\sigma(x_2,y_2,\dots,x_k,y_k,x_1,y_1) \tag{4.31}$$

Proof. It follows from Lemma 2.

Lemma 5. *The $C^{(k)}_\sigma$'s have at most simple poles in all their variables and are such that:*

$$C^{(k)}_\sigma(x_1,y_1,x_2,y_2,\dots,x_k,y_k) = \sum_{j=1}^{k-1}\frac{1}{y_k-y_j}\mathop{\mathrm{Res}}_{y'_k\to y_j} C^{(k)}_\sigma(x_1,y_1,x_2,y_2,\dots,x_k,y'_k)dy'_k \tag{4.32}$$

Proof. If σ is planar, the pair (y_k,y_j) can appear in at most one factor of eq. (4.13), and the results follows from Lemma 3.

Theorem 4. *The $C^{(k)}_\sigma$'s, with $\sigma(1)\neq k$, satisfy the recursion formula eq. (3.2):*

$$C^{(k)}_\sigma = \frac{1}{x_{\sigma^{-1}(k)}-x_1}\sum_{j=1}^{k-1}\sum_{\tau\in\Sigma(1,\dots,j)}\sum_{\rho\in\Sigma(j+1,\dots,k)}\delta_{\sigma,\tau\rho}\frac{1}{y_k-y_j}C^{(j)}_\tau C^{(k-j)}_\rho\,. \tag{4.33}$$

Proof. Let π be the cycle of length $l+1\geq 2$ of $S\sigma$ which contains x_1 and y_k:

$$\pi = (x_1\to y_{i_1-1}\to x_{i_1}\to y_{i_2-1}\to\cdots\to y_{i_{l-1}}\to x_{i_l}\to y_k\to x_1) \tag{4.34}$$

$$i_{j+1} := \sigma(i_j)+1\,,\qquad i_0 := 1\,. \tag{4.35}$$

Planarity implies that:

$$i_0 < i_1 < i_2 < \cdots < i_l\,. \tag{4.36}$$

Since σ is planar, there exists a unique way of factorizing σ as:

$$\sigma = \prod_{j=0}^{l}\sigma_j\,,\qquad \sigma_j\in\overline{\Sigma}(i_j,\dots,i_{j+1}-1)\,. \tag{4.37}$$

From the definition of $C_\sigma^{(k)}$ we have:

$$C_\sigma^{(k)} = F^{(l+1)}(x_1, y_{i_1-1}, x_{i_1}, y_{i_2-1}, \dots, y_{i_l-1}, x_{i_l}, y_k) \prod_j C_{\sigma_j}^{(i_{j+1}-i_j)}(x_{i_j}, \dots, y_{i_{j+1}-1}) \quad (4.38)$$

and using eq. (4.5), we have:

$$\begin{aligned} C_\sigma^{(k)} &= F^{(l+1)}(x_1, y_{i_1-1}, x_{i_1}, y_{i_2-1}, \dots, y_{i_l-1}, x_{i_l}, y_k) \prod_j C_{\sigma_j}^{(i_{j+1}-i_j)}(x_{i_j}, \dots, y_{i_{j+1}-1}) \\ &= \sum_{m=1}^{l} \frac{F^{(m)}(x_1, y_{i_1-1}, x_{i_1}, \dots, x_{i_{m-1}}, y_{i_m-1}) F^{(l+1-m)}(x_{i_m}, y_{i_{m+1}-1}, x_{i_{m+1}}, \dots, x_{i_l}, y_k)}{(x_{i_l}-x_1)(y_k - y_{i_m-1})} \times \\ &\quad \times \prod_j C_{\sigma_j}^{(i_{j+1}-i_j)}(x_{i_j}, \dots, y_{i_{j+1}-1}) \end{aligned} \quad (4.39)$$

notice that $i_l = \sigma^{-1}(k)$, and note:

$$\tau_m := \prod_{j=1}^{m-1} \sigma_j \,, \qquad \rho_m := \prod_{j=m}^{l} \sigma_j \,. \quad (4.40)$$

We have:

$$\tau_m \in \overline{\Sigma}(1, \dots, i_m - 1) \,, \qquad \rho_m \in \overline{\Sigma}(i_m, \dots, k) \quad (4.41)$$

Eq. (4.38) gives:

$$C_\sigma^{(k)} = \frac{1}{x_{\sigma^{-1}(k)} - x_1} \sum_{m=1}^{l} \frac{1}{y_k - y_{i_m-1}} C_{\tau_m}^{(i_m)} C_{\rho_m}^{(k-i_m)} \,. \quad (4.42)$$

It is clear, from the planarity condition that if there exists some j and τ and ρ, such that:

$$\sigma = \tau\rho \,, \qquad \tau \in \overline{\Sigma}(1, \dots, j) \,, \qquad \rho \in \overline{\Sigma}(j+1, \dots, k) \quad (4.43)$$

then, one must have $j = i_m$, $\tau = \tau_m$ and $\rho = \rho_m$ for some m.

Lemma 6. *For any transposition (k, j) (with $k \neq j$), we have:*

$$\operatorname*{Res}_{y'_k \to y_j} C_\sigma^{(k)}(x_1, y_1, x_2, y_2, \dots, x_k, y'_k) dy'_k = - \operatorname*{Res}_{y'_k \to y_j} C_{(k,j)\circ\sigma}^{(k)}(x_1, y_1, x_2, y_2, \dots, x_k, y'_k) dy'_k \quad (4.44)$$

Proof. It is trivial if y_j and y_k are not in the same face, because both sides vanish: the l.h.s. has no pole and the r.h.s. is a non-planar permutation. The case where y_j and y_k belong to the same face reduces to proving the Lemma for $\sigma = Id$.

For $\sigma = Id$, we prove it by recursion on k. It clearly works for $k = 1$ and $k = 2$. Assume that it works up to $k - 1$.

From the definition eq. (3) we have:

$$\begin{aligned} & C_{(k,j)}^{(k)}(x_1, y_1, x_2, y_2, \dots, x_k, y_k) = \\ &\quad = F^{(j)}(x_1, y_1, \dots, x_j, y_k) F^{(k-j)}(x_{j+1}, y_{j+1}, \dots, y_{k-1}, x_k, y_j) \times \\ &\quad \times F^{(2)}(x_j, y_k, x_k, y_j) \\ &\quad = \frac{F^{(j)}(x_1, y_1, \dots, x_j, y_k) F^{(k-j)}(x_{j+1}, y_{j+1}, \dots, y_{k-1}, x_k, y_j)}{(x_j - x_k)(y_k - y_j)} \end{aligned} \quad (4.45)$$

and thus:

$$\operatorname*{Res}_{y'_k\to y_j} C^{(k)}_{(k,j)}(x_1,y_1,\dots,x_k,y_k) = \frac{F^{(j)}(x_1,y_1,\dots,x_j,y_j)F^{(k-j)}(x_{j+1},y_{j+1},\dots,x_k,y_j)}{(x_j-x_k)}\,. \tag{4.46}$$

On the l.h.s. , we have from eq. (4.5):

$$\begin{aligned}
&\operatorname*{Res}_{y'_k\to y_j} F^{(k)}(x_1,y_1,x_2,y_2,\dots,x_k,y'_k) = \\
&\quad= \operatorname*{Res}_{y'_k\to y_j}\sum_{l=1}^{k-1}\frac{F^{(l)}(x_1,y_1,x_2,\dots,x_l,y_l)F^{(k-l)}(x_{l+1},y_{l+1},\dots,x_k,y'_k)}{(x_k-x_1)(y'_k-y_l)} \\
&\quad= \frac{F^{(j)}(x_1,y_1,x_2,\dots,x_j,y_j)F^{(k-j)}(x_{j+1},y_{j+1},\dots,x_k,y_j)}{(x_k-x_1)} + \\
&\qquad+\sum_{l=1}^{j-1}\frac{F^{(l)}(x_1,y_1,x_2,\dots,x_l,y_l)}{(x_k-x_1)(y_j-y_l)}\operatorname*{Res}_{y'_k\to y_j} F^{(k-l)}(x_{l+1},y_{l+1},\dots,x_k,y'_k)\,.
\end{aligned} \tag{4.47}$$

Then, from the recursion hypothesis, and from eq. (4.46) we have:

$$\begin{aligned}
&\operatorname*{Res}_{y'_k\to y_j} F^{(k)}(x_1,y_1,x_2,y_2,\dots,x_k,y'_k) = \\
&\quad= \frac{F^{(j)}(x_1,y_1,x_2,\dots,x_j,y_j)F^{(k-j)}(x_{j+1},y_{j+1},\dots,x_k,y_j)}{(x_k-x_1)} - \\
&\qquad-\sum_{l=1}^{j-1}\frac{F^{(l)}(x_1,y_1,x_2,\dots,x_l,y_l)}{(x_k-x_1)(y_j-y_l)}\operatorname*{Res}_{y'_k\to y_j} C^{(k-l)}_{(k,j)}(x_{l+1},y_{l+1},\dots,x_k,y'_k) \\
&\quad= \frac{F^{(j)}(x_1,y_1,x_2,\dots,x_j,y_j)F^{(k-j)}(x_{j+1},y_{j+1},\dots,x_k,y_j)}{(x_k-x_1)} - \\
&\qquad-\sum_{l=1}^{j-1}\frac{F^{(l)}(x_1,y_1,x_2,\dots,x_l,y_l)}{(x_k-x_1)(y_j-y_l)}\times \\
&\qquad\times\frac{F^{(j-l)}(x_{l+1},y_{l+1},\dots,x_j,y_j)F^{(k-j)}(x_{j+1},y_{j+1},\dots,y_{k-1},x_k,y_j)}{(x_j-x_k)}\,.
\end{aligned} \tag{4.48}$$

In the last line, we use again eq. (4.5) and get:

$$\begin{aligned}
&\operatorname*{Res}_{y'_k\to y_j} F^{(k)}(x_1,y_1,x_2,y_2,\dots,x_k,y'_k) = \\
&\quad= \frac{F^{(j)}(x_1,y_1,x_2,\dots,x_j,y_j)F^{(k-j)}(x_{j+1},y_{j+1},\dots,x_k,y_j)}{(x_k-x_1)} - \\
&\qquad-(x_j-x_1)\frac{F^{(j)}(x_1,y_1,x_2,\dots,x_j,y_j)}{(x_k-x_1)}\frac{F^{(k-j)}(x_{j+1},y_{j+1},\dots,y_{k-1},x_k,y_j)}{(x_j-x_k)}\,.
\end{aligned} \tag{4.49}$$

One can then see that the sum of eq. (4.46) and eq. (4.49) vanishes, and the recursion hypothesis is proven for k.

Lemma 7.

$$\forall\, k>1\,,\qquad \sum_{\sigma\in\overline{\Sigma}_k} C^{(k)}_\sigma(x_1,y_1,x_2,y_2,\dots,x_k,y_k)=0\,. \tag{4.50}$$

Proof. This expression is a rational function of all its variables. Consider the poles at $y_k = y_j$, and write $\tau = (k, j)$. One can split the symmetric group $\overline{\Sigma}_k$ into its two conjugacy classes wrt the subgroup generated by τ: $\overline{\Sigma}_k = [Id] \oplus [\tau]$. In other words:

$$\sum_{\sigma\in\overline{\Sigma}_k} C_\sigma(x_1, y_1, x_2, y_2, \ldots, x_k, y_k) = \sum_{\sigma\in\overline{\Sigma}_k/\tau} C_\sigma(x_1, y_1, x_2, y_2, \ldots, x_k, y_k) + \\ + C_{\tau\sigma}(x_1, y_1, x_2, y_2, \ldots, x_k, y_k)\,. \tag{4.51}$$

From Lemma 6, the terms in the r.h.s. have no pole at $y_k = y_j$. Similarly, using cyclicity and doing the same for the x's, we prove the lemma.

The Lemmas we have just proven, are sufficient to prove the main theorem 3. This is done in section 4.7.

4.6 Computation of the rational functions $F^{(k)}(x_1, y_1, \ldots, x_k, y_k)$

Although the exact computation of the $F^{(k)}$'s is not necessary for proving theorem 3, we do it for completeness. In this section we give an explicit (and non-recursive) formula for the $F^{(k)}$'s.

A practical way of computing these formulas is described in appendix A.

Definition 4. *To every permutation $\sigma \in \overline{\Sigma}_{k-1}$, we associate a weight f_σ computed as follows:*

$$f_\sigma = \prod_{n=1}^{l} \prod_{j=2}^{l_n} g_{i_{n,1}, i_{n,j}, i_{n,j+1}} \prod_{n=2}^{\tilde{l}} \prod_{j=2}^{\tilde{l_n}} g_{\tilde{i}_{n,j}, \tilde{i}_{n,1}, \sigma(\tilde{i}_{n,j})} \prod_{j=1}^{\tilde{l_1}} g_{\tilde{i}_{1,j}, k, \sigma(\tilde{i}_{1,j})} \tag{4.52}$$

where $g_{i,h,j}$ is defined as follows:

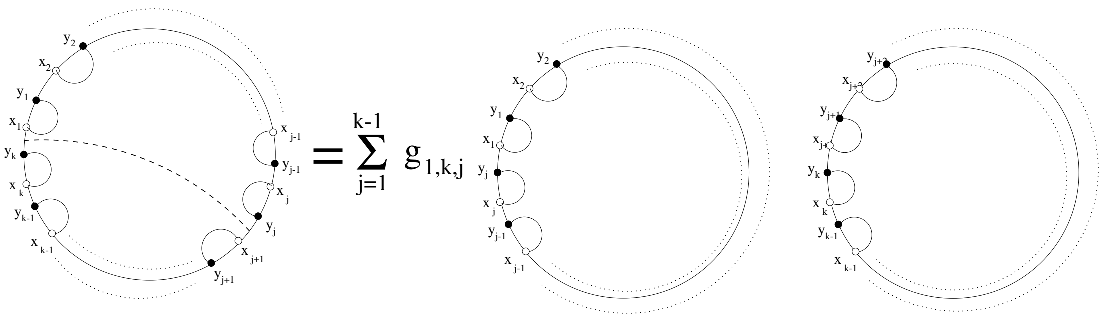
$$g_{i,h,j} := \frac{1}{x_h - x_i} \frac{1}{y_h - y_j} \tag{4.53}$$

and σ and $S\sigma$ are decomposed into their product of cycles as in Definition 3.

Theorem 5. *$F^{(k)}(x_1, y_1, \ldots, x_k, y_k)$ is obtained as the sum of the weights f_σ's over all $\sigma \in \Sigma_{k-1}$:*

$$F^{(k)}(x_1, y_1, \ldots, x_k, y_k) = \sum_{\sigma\in\Sigma_{k-1}} f_\sigma\,. \tag{4.54}$$

Proof. First of all, let us interpret diagrammatically the recursion relation eq. (4.5) defining the F's:



(4.55)

Actually, this recursion relation is nothing else but a rule for cutting a graph along the dashed line into two smaller ones. The weight of a graph is then obtained as the sum over all the possible ways of cutting it in two.

Notice that $F^{(k)}$ is the sum of $\mathrm{Cat}\,(k-1)$ different terms.

Let us now explicit this bijection with the graphs with $k-1$ arches. In order to compute one of the terms composing $F^{(k)}$, one has to cut it with the help of the recursion relation until one obtains only graphs with one arch. That is to say that one cuts it $k-1$ times along non intersecting lines (corresponding to the dashed one in the recursion relation). If one draws these cutting lines on the circle, one obtains a graph with $k-1$ arches dual of the original one. Thus every way of cutting a graph with $k-1$ arches is associated to a planar permutation $\sigma \in \overline{\Sigma}(i \dots k-1)$. Let us now prove that the term obtained by this cutting is equal to f_σ.

For the sake of simplicity, one denotes the identity graph of $(x_j, y_j, \dots, x_k, y_k)$ by circle $(j, j+1, \dots, k)$. In these conditions, the recursion relation reads:

$$(1,2,\dots,k) = \sum_{j=1}^{k-1} g_{1,k,j}(1,\dots,j)(j+1,\dots,k)\,. \tag{4.56}$$

Let σ be a permutation of $(1,\dots,k-1)$. Cut it along the line going from the boundary (x_1, y_k) to $(y_{\sigma(1)}, x_{S\sigma(1)})$. It results from this operation the factor $g_{1,k,\sigma(1)}$ and the circles $(1,\dots,\sigma(1))$ and $(S\sigma(1),\dots,k)$:

$$(1,\dots,k) \to^\sigma g_{1,k,\sigma(1)}(1,\dots,\sigma(1))(S\sigma(1),\dots,k) \tag{4.57}$$

Then cutting the circle $(S\sigma(1),\dots,k)$ along $(y_k, x_{S\sigma(1)}) \to (y_{\sigma S\sigma(1)}, x_{S\sigma S\sigma(1)})$ gives:

$$(S\sigma(1),\dots,k) \to^\sigma g_{S\sigma(1),k,\sigma S\sigma(1)}(S\sigma(1),\dots,\sigma S\sigma(1))(S\sigma S\sigma(1),\dots,k) \tag{4.58}$$

One pursues this procedure step by step by always cutting the circle containing k. Using the former notations, this reads:

$$(1,\dots,k) \to^\sigma \prod_{j=1}^{\tilde{l}_1} g_{\tilde{i}_{1,j},k,\sigma(\tilde{i}_{1,j})}(\tilde{i}_{1,j},\dots,\sigma(\tilde{i}_{1,j}))\,. \tag{4.59}$$

So one has computed the weight associated to the first $S\sigma$ - cycle. The remaining circles correspond to σ-cycles. Let us compute their weight by considering for example $(\tilde{i}_{1,1},\dots,\sigma(\tilde{i}_{1,1})) = (i_{1,1},\dots,i_{1,2})$.

The cut along the line $(x_{i_{1,1}}, y_{i_{1,2}}) \to (y_{i_{1,3}}, x_{S(i_{1,1})})$ gives:

$$(i_{1,1},\dots,i_{1,2}) \to^\sigma g_{i_{1,1},i_{1,2},i_{1,3}}(i_{1,1},\dots,i_{1,3})(S(i_{1,3}),\dots,i_{1,2})\,. \tag{4.60}$$

Keeping on cutting the circle containing $i_{1,1}$ at every step gives:

$$(i_{1,1},\dots,i_{1,2}) \to^\sigma \prod_{j=2}^{l_1} g_{i_{1,1},i_{1,j},i_{1,j+1}}(S(i_{1,j+1}),\dots,i_{1,j})\,. \tag{4.61}$$

JHEP08(2005)028

One can notice that the remaining circles in the r.h.s. correspond to cycles of $S\sigma$ whose contribution has not been taken into account yet. One can then compute their values by following a procedure similar to the one used for the first $S\sigma$-cycle.

One can then recursively cut the circles so that one finally obtains only circles containing only one element. This recursion is performed by alternatively processing on σ-cycles and $S\sigma$-cycles.

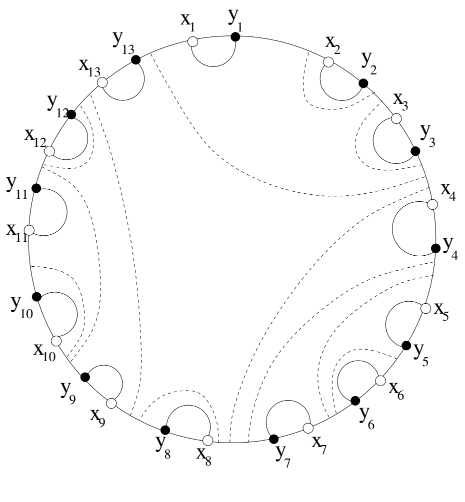
Thus, one straightforwardly finds:

$$(1,\ldots,k) \to^{\sigma} \prod_{n=1}^{l}\prod_{j=2}^{l_n} g_{i_{n,1},i_{n,j},i_{n,j+1}} \prod_{n=2}^{\tilde l}\prod_{j=2}^{\tilde l_n} g_{\tilde i_{n,j},\tilde i_{n,1},\sigma(\tilde i_{n,j})} \prod_{j=1}^{\tilde l_1} g_{\tilde i_{1,j},k,\sigma(\tilde i_{1,j})} = f_\sigma \,. \tag{4.62}$$

And then:

$$F^{(k)} = \sum_{\sigma\in\overline{\Sigma}_{k-1}} f_\sigma \tag{4.63}$$

Example: let us compute the weight associated to the permutation $\sigma \in \overline{\Sigma}_{12}$ introduced earlier. Starting from the circle $(1,\ldots,13)$, one will proceed step by step the following cutting:

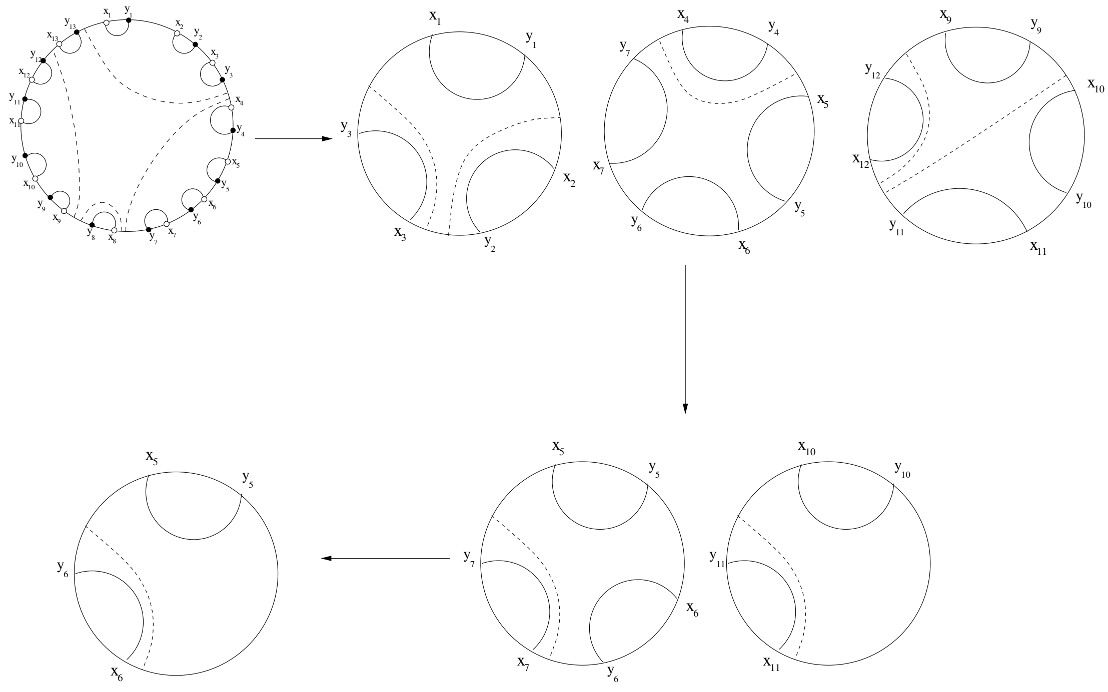


(4.64)

The first step consists in cutting along the $\tilde\sigma_1$ cycle. The dashed lines show where one cuts the circles. Note that one do not represent the circles of unite length. The associated weight is $g_{1,13,3}\, g_{4,13,7}\, g_{8,13,8}\, g_{9,13,12}$.

The second step consists in cutting along the remaining σ cycles. One associates the weight $g_{1,3,2}\, g_{1,2,1} \times g_{4,7,4} \times g_{9,12,11}\, g_{9,11,9}$ to this step.

The weights associated to the two last cuttings are $g_{5,7,6} \times g_{10,11,10}$ and $g_{5,6,5}$.



(4.65)

Finally, the weight of this planar permutation is then:

$$f_\sigma = g_{1,13,3}\, g_{4,13,7}\, g_{8,13,8}\, g_{9,13,12}\, g_{1,3,2}\, g_{1,2,1}\, g_{4,7,4}\, g_{9,12,11}\, g_{9,11,9}\, g_{5,7,6}\, g_{10,11,10}\, g_{5,6,5}\,. \tag{4.66}$$

4.7 Proof of the main theorem

We now prove that the function $\widehat{W}$ defined by the r.h.s. of eq. (3.1) and the functions $\widehat{U}$ and $\widehat{P}$ defined in theorem 2 satisfy the system of equations eq. (2.30) and eq. (2.31).

Proof of theorem 2: using eq. (4.32), one has:

$$\begin{aligned}
\widehat{U}_k(x_1,\dots,y_k) &= \mathop{\mathrm{Pol}}_{y_k} V_2'(y_k)\, \widehat{W}_k(x_1,y_1,\dots,x_k,y_k) \\
&= \mathop{\mathrm{Pol}}_{y_k} V_2'(y_k) \sum_{\sigma\in\Sigma_k} C_\sigma^{(k)}(x_1,y_1,\dots,x_k,y_k) \prod_{i=1}^{k} W_1(x_i,y_{\sigma(i)}) \\
&= \sum_{\sigma\in\Sigma_k}\sum_{j\neq k} \mathop{\mathrm{Res}}_{y_k'\to y_j} C_\sigma^{(k)}(x_1,y_1,\dots,x_k,y_k')dy_k' \prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)},y_i)\,\times \\
&\quad \times \mathop{\mathrm{Pol}}_{y_k} \frac{V_2'(y_k) W_1(x_{\sigma^{-1}(k)},y_k)}{y_k-y_j} \\
&= \sum_{\sigma\in\Sigma_k}\sum_{j\neq k} \mathop{\mathrm{Res}}_{y_k'\to y_j} C_\sigma^{(k)}(x_1,y_1,\dots,x_k,y_k')dy_k' \prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)},y_i)\,\times \\
&\quad \times \mathop{\mathrm{Pol}}_{y_k} \frac{(\tilde{W}(y_k)-X(y_k)) W_1(x_{\sigma^{-1}(k)},y_k)}{y_k-y_j} \\
&= -\sum_{\sigma\in\Sigma_k}\sum_{j\neq k} \mathop{\mathrm{Res}}_{y_k'\to y_j} C_\sigma^{(k)}(x_1,y_1,\dots,x_k,y_k')dy_k' \prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)},y_i)\,\times
\end{aligned}$$

$$\times \operatorname*{Pol}_{y_k} \frac{X(y_k)W_1(x_{\sigma^{-1}(k)}, y_k)}{y_k - y_j}$$

$$= \sum_{\sigma\in\Sigma_k} \sum_{j\neq k} \operatorname*{Res}_{y'_k\to y_j} C^{(k)}_\sigma(x_1, y_1, \dots, x_k, y'_k) dy'_k \prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)}, y_i) \times$$

$$\times \operatorname*{Pol}_{y_k} \frac{(x_{\sigma^{-1}(k)} - X(y_k))W_1(x_{\sigma^{-1}(k)}, y_k) - (x_{\sigma^{-1}(k)} - X(y_j))W_1(x_{\sigma^{-1}(k)}, y_j)}{y_k - y_j}$$

$$= \sum_{\sigma\in\Sigma_k} \sum_{j\neq k} \operatorname*{Res}_{y'_k\to y_j} C^{(k)}_\sigma(x_1, y_1, \dots, x_k, y'_k) dy'_k \prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)}, y_i) \times \tag{4.67}$$

$$\times \frac{(x_{\sigma^{-1}(k)} - X(y_k))W_1(x_{\sigma^{-1}(k)}, y_k) - (x_{\sigma^{-1}(k)} - X(y_j))W_1(x_{\sigma^{-1}(k)}, y_j)}{y_k - y_j}.$$

Indeed, using eq. (2.26), one sees that the last expression is a polynomial in y_k:

$$\frac{(x_{\sigma^{-1}(k)} - X(y_k))W_1(x_{\sigma^{-1}(k)}, y_k) - (x_{\sigma^{-1}(k)} - X(y_j))W_1(x_{\sigma^{-1}(k)}, y_j)}{y_k - y_j} =$$

$$= \frac{\frac{E(x_{\sigma^{-1}(k)}, y_k)}{y_k - Y(x_{\sigma^{-1}(k)})} - \frac{E(x_{\sigma^{-1}(k)}, y_j)}{y_j - Y(x_{\sigma^{-1}(k)})}}{y_k - y_j}. \tag{4.68}$$

We have to check eq. (2.31), i.e. that $A = 0$ with:

$$A := \sum_j \frac{\widehat{W}_{k-j}(x_{j+1}, \dots, y_k) - \widehat{W}_{k-j}(x_{j+1}, \dots, x_k, y_j)}{y_k - y_j} \widehat{W}_j(x_1, \dots, y_j) -$$

$$-\widehat{U}_k(x_1, \dots, y_k) + (x_1 - X(y_k)) \widehat{W}_k(x_1, y_1, x_2, \dots, x_k, y_k). \tag{4.69}$$

We have:

$$A = \sum_{j\neq k} \sum_\pi \sum_\tau \frac{C^{(j)}_\tau(x_1, \dots, y_j)\, C^{(k-j)}_\pi(x_{j+1}, \dots, y_k)}{y_k - y_j} \times$$

$$\times \prod_{i=1}^{j} W_1(x_{\tau^{-1}(i)}, y_i) \prod_{i=j+1}^{k} W_1(x_{\pi^{-1}(i)}, y_i) -$$

$$- \sum_{j\neq k} \sum_\pi \sum_\tau \frac{C^{(j)}_\tau(x_1, \dots, y_j)\, C^{(k-j)}_\pi(x_{j+1}, \dots, y_j)}{y_k - y_j} W_1(x_{\pi^{-1}(k)}, y_j) \times$$

$$\times \prod_{i=1}^{j} W_1(x_{\tau^{-1}(i)}, y_i) \prod_{i=j+1}^{k-1} W_1(x_{\pi^{-1}(i)}, y_i) -$$

$$- \sum_\sigma \sum_{j\neq k} \frac{(x_{\sigma^{-1}(k)} - X(y_k))}{y_k - y_j} W_1(x_{\sigma^{-1}(k)}, y_k) \times$$

$$\times \left(\operatorname*{Res}_{y_k\to y_j} C^{(k)}_\sigma\right) \prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)}, y_i) +$$

$$+ \sum_\sigma \sum_{j\neq k} \frac{(x_{\sigma^{-1}(k)} - X(y_j))}{y_k - y_j} W_1(x_{\sigma^{-1}(k)}, y_j) \times$$

$$\times\left(\operatorname*{Res}_{y_k\to y_j} C_\sigma^{(k)}\right)\prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)},y_i)+$$

$$+(x_1-X(y_k))\sum_\sigma C_\sigma^{(k)}(x_1,\dots,y_k)W_1(x_{\sigma^{-1}(k)},y_k)\prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)},y_i)\,. \quad (4.70)$$

Using eq. (4.32) in the last line, adding it to the 4th line, and using eq. (4.32) again, we get:

$$\begin{aligned}
A =& \sum_{j\neq k}\sum_\pi\sum_\tau \frac{C_\tau^{(j)}(x_1,\dots,y_j)\,C_\pi^{(k-j)}(x_{j+1},\dots,y_k)}{y_k-y_j}\times\\
&\times\prod_{i=1}^{j} W_1(x_{\tau^{-1}(i)},y_i)\prod_{i=j+1}^{k} W_1(x_{\pi^{-1}(i)},y_i)+\\
&+(x_1-x_{\sigma^{-1}(k)})\sum_\sigma C_\sigma^{(k)}(x_1,\dots,y_k)W_1(x_{\sigma^{-1}(k)},y_k)\prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)},y_i)-\\
&-\sum_{j\neq k}\sum_\pi\sum_\tau \frac{C_\tau^{(j)}(x_1,\dots,y_j)\,C_\pi^{(k-j)}(x_{j+1},\dots,y_j)}{y_k-y_j}W_1(x_{\pi^{-1}(k)},y_j)\times\\
&\times\prod_{i=1}^{j} W_1(x_{\tau^{-1}(i)},y_i)\prod_{i=j+1}^{k-1} W_1(x_{\pi^{-1}(i)},y_i)+\\
&+\sum_\sigma\sum_{j\neq k}\frac{(x_{\sigma^{-1}(k)}-X(y_j))}{y_k-y_j}W_1(x_{\sigma^{-1}(k)},y_j)\times\\
&\times\left(\operatorname*{Res}_{y_k\to y_j} C_\sigma^{(k)}\right)\prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)},y_i)\,.
\end{aligned}\quad (4.71)$$

Using eq. (3.2) in the second line, exactly cancels the first line, and thus we get:

$$\begin{aligned}
A =& -\sum_{j\neq k}\sum_\pi\sum_\tau \frac{C_\tau^{(j)}(x_1,\dots,y_j)\,C_\pi^{(k-j)}(x_{j+1},\dots,y_j)}{y_k-y_j}W_1(x_{\pi^{-1}(k)},y_j)\times\\
&\times\prod_{i=1}^{j} W_1(x_{\tau^{-1}(i)},y_i)\prod_{i=j+1}^{k-1} W_1(x_{\pi^{-1}(i)},y_i)+\\
&+\sum_\sigma\sum_{j\neq k}\frac{(x_{\sigma^{-1}(k)}-X(y_j))}{y_k-y_j}W_1(x_{\sigma^{-1}(k)},y_j)\times\\
&\times\left(\operatorname*{Res}_{y_k\to y_j} C_\sigma^{(k)}\right)\prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)},y_i)
\end{aligned}\quad (4.72)$$

which is a rational fraction in y_k with poles at $y_k=y_j$ for some j. From Lemma 6, A as defined in eq. (4.69) cannot have poles at $y_k=y_j$, thus $A=0$.

Now, we have to check eq. (2.30) Using eq. (4.32), one has:

$$\widehat{P}_k(x_1,\dots,y_k)-\widehat{W}_{k-1}(x_2,\dots,x_k,y_1)=$$

$$
\begin{aligned}
&= \operatorname*{Pol}_{x_1} V_1'(x_1)\, \widehat{U}_k(x_1, y_1, \dots, x_k, y_k) \\
&= \operatorname*{Pol}_{x_1} Y(x_1) \sum_{\sigma \in \Sigma_k} \sum_{j \neq k} \operatorname*{Res}_{y_k' \to y_j} C_\sigma^{(k)} \prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)}, y_i) \times \\
&\quad \times \frac{U_1(x_{\sigma^{-1}(k)}, y_k) - U_1(x_{\sigma^{-1}(k)}, y_j)}{y_k - y_j} \\
&= \operatorname*{Pol}_{x_1} Y(x_1) \sum_{\sigma \in \Sigma_k,\, \sigma(1)=k} \sum_{j \neq k} \operatorname*{Res}_{y_k' \to y_j} C_\sigma^{(k)} \prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)}, y_i) \times \\
&\quad \times \frac{U_1(x_1, y_k) - U_1(x_1, y_j)}{y_k - y_j} + \\
&\quad + \operatorname*{Pol}_{x_1} Y(x_1) \sum_{\sigma \in \Sigma_k,\, \sigma(1)\neq k} \sum_{j \neq k} \operatorname*{Res}_{y_k' \to y_j} C_\sigma^{(k)}\, W_1(x_1, y_{\sigma(1)}) \prod_{i \neq k, \sigma(1)} W_1(x_{\sigma^{-1}(i)}, y_i) \times \\
&\quad \times \frac{U_1(x_{\sigma^{-1}(k)}, y_k) - U_1(x_{\sigma^{-1}(k)}, y_j)}{y_k - y_j} \\
&= - \sum_{\sigma \in \Sigma_k,\, \sigma(1)=k} \sum_{j \neq k} \sum_{l \neq 1} \operatorname*{Res}_{x_1 \to x_l} \operatorname*{Res}_{y_k' \to y_j} C_\sigma^{(k)} \prod_{i=1}^{k-1} W_1(x_{\sigma^{-1}(i)}, y_i) \times \\
&\quad \times \frac{E(x_1, y_k) - E(x_1, y_j) - E(x_l, y_k) + E(x_l, y_j)}{(x_1 - x_l)(y_k - y_j)} + \\
&\quad + \sum_{\sigma \in \Sigma_k,\, \sigma(1)\neq k} \sum_{j \neq k} \sum_{l \neq 1} \operatorname*{Res}_{x_1 \to x_l} \operatorname*{Res}_{y_k' \to y_j} C_\sigma^{(k)} \prod_{i \neq k, \sigma(1)} W_1(x_{\sigma^{-1}(i)}, y_i) \times \\
&\quad \times \frac{(y_{\sigma(1)} - Y(x_1)) W_1(x_1, y_{\sigma(1)}) - (y_{\sigma(1)} - Y(x_l)) W_1(x_l, y_{\sigma(1)})}{x_1 - x_l} \times \\
&\quad \times \frac{U_1(x_{\sigma^{-1}(k)}, y_k) - U_1(x_{\sigma^{-1}(k)}, y_j)}{y_k - y_j}\,.
\end{aligned}
\tag{4.73}
$$

In order to satisfy eq. (2.30), we must prove that $B = 0$, where:

$$
\begin{aligned}
B := \sum_{l=2}^{k} & \frac{\widehat{W}_{l-1}(x_1, y_1, \dots, y_{l-1}) - \widehat{W}_{l-1}(x_l, y_1, x_2, \dots, y_{l-1})}{x_1 - x_l} \times \\
& \times \widehat{U}_{k-l+1}(x_l, y_l, \dots, x_k, y_k) + \\
& + \sum_{l=1}^{k-1} \widehat{W}_l(x_1, \dots, y_l)\, \widehat{W}_{k-l}(x_k, y_l, \dots, y_{k-1}) - \\
& - \widehat{P}_k(x_1, y_1, x_2, \dots, x_k, y_k) - \\
& - (Y(x_1) - y_k)\, \widehat{U}_k(x_1, \dots, y_k)\,.
\end{aligned}
\tag{4.74}
$$

One does it in a way very similar to the previous one, i.e. first prove, using eq. (3.2), that B is a rational fraction of x_1, with poles at $x_1 = x_l$. But B can have no pole at $x_1 = x_l$, so,$B = 0$.

5. Matrix form of correlation functions

So far, we have computed mixed correlation functions with only one trace, i.e. the generating function of connected discrete surfaces with one boundary. In this section, we generalize this theory to the computation of generating functions of non-connected discrete surfaces with any number of boundaries. In order to derive those correlation functions, a matrix approach of the problem, similar to the one developed in [9], is used.

Definition 5. *Let k be a positive integer. Let π and π' be two permutations of Σ_k and decompose ${\pi'}^{-1}\pi$ into the product of its irreducible cycles:*

$$ {\pi'}^{-1}\pi = P_1 P_2 \dots P_n \tag{5.1} $$

Each cycle P_i of ${\pi'}^{-1}\pi$, of length p_i, is denoted:

$$ P_m = (i_{m,1} \rightarrow^{\pi} j_{m,1} \rightsquigarrow^{{\pi'}^{-1}} i_{m,2} \rightarrow^{\pi} j_{m,2} \rightsquigarrow^{{\pi'}^{-1}} \dots \rightsquigarrow^{{\pi'}^{-1}} i_{m,p_m} \rightarrow^{\pi} j_{m,p_m} \rightsquigarrow^{{\pi'}^{-1}} i_{m,1})\,. \tag{5.2} $$

For any $(x_1, y_1, x_2, y_2, \dots, x_k, y_k) \in \mathbf{C}$, we define:

$$ \mathcal{W}^k_{\pi,\pi'}(x_1, y_1, \dots, x_k, y_k) := \left\langle \prod_{m=1}^{n} \left(\delta_{p_m,1} + \frac{1}{N} \operatorname{Tr} \prod_{j=1}^{p_m} \frac{1}{(M_1 - x_{i_{m,j}})(M_2 - y_{j_{m,j}})} \right) \right\rangle \tag{5.3} $$

which is a $k! \times k!$ matrix.

Let us now generalize the notion of planarity of a permutation.

Definition 6. *Let k be a positive integer. Let π and π' be two permutations of Σ_k.*

A permutation $\sigma \in \Sigma_k$ is said to be planar wrt (π, π') if

$$ n_{\text{cycles}}(\pi^{-1}\sigma) + n_{\text{cycles}}({\pi'}^{-1}\sigma) = k + n_{\text{cycles}}({\pi'}^{-1}\pi)\,. \tag{5.4} $$

Let $\Sigma_k^{(\pi,\pi')} \subset \Sigma_k$ be the set of permutations planar wrt (π, π').

Graphically, if one draws the sets of points $(x_{i_{1,1}}, y_{j_{1,1}}, x_{i_{1,2}}, y_{j_{1,2}}, \dots, x_{i_{1,p_1}}, y_{j_{1,p_1}})$, $(x_{i_{2,1}}, y_{j_{2,1}}, x_{i_{2,2}}, y_{j_{2,2}}, \dots, x_{i_{2,p_2}}, y_{j_{2,p_2}})$, ... , $(x_{i_{p,1}}, y_{j_{p,1}}, x_{i_{p,2}}, y_{j_{p,2}}, \dots, x_{i_{n,p_n}}, y_{j_{n,p_n}})$ on n circles and link each pair $(x_j, y_{\sigma(j)})$ by a line, these lines do not intersect nor go from one circle to another.

Remark 1. One can straightforwardly see two properties of these sets:

- This relation of planarity wrt to (π, π') is symmetric in π and π', that is to say:

$$ \Sigma_k^{(\pi,\pi')} = \Sigma_k^{(\pi',\pi)}\,. \tag{5.5} $$

- The planarity defined in eq. (1) corresponds to $\pi = Id$ and $\pi' = S^{-1}$:

$$ \overline{\Sigma}_k = \Sigma_k^{(Id,S^{-1})}\,. \tag{5.6} $$

Directly from these definitions and the preceding results comes the following theorem computing any generating function of discrete surface with boundaries.

Theorem 6.

$$\boxed{\mathcal{W}^k_{\pi,\pi'}(x_1,y_1,x_2,y_2,\ldots,x_k,y_k) = \sum_\sigma \mathcal{C}^k_{\sigma,\pi,\pi'}(x_1,y_1,\ldots,x_k,y_k)\prod_{i=1}^k W_1(x_i,y_{\sigma(i)})} \tag{5.7}$$

where $\mathcal{C}^k_\sigma$ is the $k! \times k!$-matrix defined by:

- $\mathcal{C}^k_{\sigma,\pi,\pi'}(x_1,y_1,x_2,y_2,\ldots,x_k,y_k) := 0$ *if σ is not planar wrt (π,π');*
- *if σ is planar wrt (π,π'):*

$$\begin{aligned}
&\mathcal{C}^k_{\sigma,\pi,\pi'}(x_1,y_1,\ldots,x_k,y_k) := \\
&\quad := \prod_{m=1}^{a} F^{(a_m)}(x_{r_{m,1}},y_{\sigma(r_{m,1})},x_{r_{m,2}},y_{\sigma(r_{m,2})},\ldots,x_{r_{m,a_m}},y_{\sigma(r_{m,a_m})}) \times \\
&\quad\quad \times \prod_{m=1}^{\tilde a} F^{(\tilde a_m)}(x_{\tilde r_{m,1}},y_{\sigma(\tilde r_{m,1})},x_{\tilde r_{m,2}},y_{\sigma(\tilde r_{m,2})},\ldots,x_{\tilde r_{m,a_m}},y_{\sigma(\tilde r_{m,a_m})}) .
\end{aligned} \tag{5.8}$$

with the decompositions of $\pi^{-1}\sigma$ and $\pi'^{-1}\sigma$ into their products of cycles:

$$\pi^{-1}\sigma = \pi_1\pi_2\ldots\pi_a\,, \qquad \pi'^{-1}\sigma = \tilde\pi_1\tilde\pi_2\ldots\tilde\pi_{\tilde a} \tag{5.9}$$

such that:

$$\pi_m = (r_{m,1},r_{m,2},\ldots,r_{m,a_m})\,, \qquad \tilde\pi_m = (\tilde r_{m,1},\tilde r_{m,2},\ldots,\tilde r_{m,\tilde a_m})\,. \tag{5.10}$$

Remark 2. From the definition, one can see that $\sigma(r_{m,a_m}) = \pi(r_m,1)$ and $\sigma(\tilde r_{m,a_m}) = \pi'(\tilde r_m,1)$ for any m.

5.1 Properties of the C^k_σ's.

Lemma 8. *The matrices $\mathcal{C}^k_\sigma$ are symmetric.*

Proof. It comes directly from the definition.

Lemma 9.

$$\sum_\sigma \mathcal{C}^k_\sigma = Id \tag{5.11}$$

Proof. One has:

$$\mathcal{W}^k_{\pi,\pi'}(x_1,y_1,x_2,y_2,\ldots,x_k,y_k) = \sum_\sigma \mathcal{C}^k_{\sigma,\pi,\pi'}\prod_{i=1}^k W_1(x_i,y_{\sigma(i)})\,. \tag{5.12}$$

Let us shift all the x's by a translation a and send $a\to\infty$, i.e. replace all the x_i's by x_i+a. In the l.h.s. , only the δ-terms of Definition 5 survive in the limit $a\to\infty$, and thus the l.h.s. tends towards the identity matrix. In the r.h.s. , notice that $W_1(x_i+a,y_{\sigma(i)})\to 1$. And $\mathcal{C}^k_\sigma$, which depends only on the differences between x_i's, is independent of a.

5.2 Some commutation properties

Definition 7. *Let $\mathcal{M}^k(\vec{x},\vec{y},\xi,\eta)$ be the $k!\times k!$ matrix defined by:*

$$\mathcal{M}^k(\vec{x},\vec{y},\xi,\eta)_{\pi,\pi'} := \prod_i (\delta_{\pi(i),\pi'(i)} + \frac{1}{(\xi-x_i)(\eta-y_{\pi(i)})}) . \tag{5.13}$$

Let $\mathcal{A}^{(k)}(x_1,y_1,\dots,x_k,y_k)$ be the $k!\times k!$ matrix defined by:

$$\begin{cases} \mathcal{A}^{(k)}_{\pi,\pi}(x_1,y_1,\dots,x_k,y_k) := \sum_i x_i y_{\pi(i)} \\ \mathcal{A}^{(k)}_{\pi,\pi'}(x_1,y_1,\dots,x_k,y_k) := 1 \quad \text{if } \pi\pi'^{-1} = \text{transposition} \\ \mathcal{A}^{(k)}_{\pi,\pi'}(x_1,y_1,\dots,x_k,y_k) := 0 \text{ otherwise} \end{cases} \tag{5.14}$$

Theorem 7.

$$\boxed{\forall\sigma,\xi,\eta\,, \qquad [\mathcal{M}^k(\vec{x},\vec{y},\xi,\eta),\mathcal{C}^k_\sigma(\vec{x},\vec{y})]=0} \tag{5.15}$$

and

$$\boxed{\forall\xi,\eta\,, \qquad [\mathcal{M}^k(\vec{x},\vec{y},\xi,\eta),\mathcal{W}^k(\vec{x},\vec{y})]=0} \tag{5.16}$$

Proof. Let us define:

$$\widetilde{\mathcal{M}}(\vec{x},\vec{y},\xi,\eta) := \mathcal{M}(N\vec{x},\vec{y},N\xi,\eta) \tag{5.17}$$

and

$$\widetilde{\mathcal{W}}^k_{\pi,\pi'}(x_1,y_1,\dots,x_k,y_k) := \left\langle \prod_{m=1}^n \left(\delta_{p_m,1} + \operatorname{Tr}\prod_{j=1}^{p_m} \frac{1}{N}\frac{1}{(M_1-x_{i_{m,j}})(M_2-y_{j_{m,j}})} \right) \right\rangle . \tag{5.18}$$

It was proven in [9] that:

$$[\widetilde{\mathcal{M}}^k(\vec{x},\vec{y},\xi,\eta),\widetilde{\mathcal{W}}^k(\vec{x},\vec{y})]=0 . \tag{5.19}$$

Now, in the large-N limit, the factorization property [3] $\langle \operatorname{Tr}\operatorname{Tr}\rangle \sim \langle\operatorname{Tr}\rangle\langle\operatorname{Tr}\rangle$, implies:

$$\begin{aligned} \widetilde{\mathcal{W}}^k_{\pi,\pi'}(x_1,y_1,x_2,y_2,\dots,x_k,y_k) &\sim N^{n_{\text{cycles}}(\pi'^{-1}\pi)-k} \times \\ &\times \prod_{m=1}^n W_{p_m}(x_{i_{m,1}},y_{\pi(i_{m,1})},\dots,x_{i_{m,p_m}},x_{\pi(i_{m,p_m})}) \\ &\sim N^{n_{\text{cycles}}(\pi'^{-1}\pi)-k}\mathcal{W}^k_{\pi,\pi'}(x_1,y_1,x_2,y_2,\dots,x_k,y_k) \end{aligned} \tag{5.20}$$

and using theorem 6, we have:

$$\begin{aligned} &\widetilde{\mathcal{W}}^k_{\pi,\pi'}(x_1,y_1,x_2,y_2,\dots,x_k,y_k) \sim \\ &\quad \sim N^{n_{\text{cycles}}(\pi'^{-1}\pi)-k} \sum_\sigma \mathcal{C}^k_{\sigma,\pi,\pi'}(x_1,y_1,x_2,y_2,\dots,x_k,y_k) \prod_{i=1}^k W_1(x_i,y_{\sigma(i)}) . \end{aligned} \tag{5.21}$$

Notice that

$$\begin{aligned} &\mathcal{C}^k_{\sigma,\pi,\pi'}(x_1,y_1,x_2,y_2,\dots,x_k,y_k) = \\ &\quad = N^{k-n_{\text{cycles}}(\pi^{-1}\sigma)+k-n_{\text{cycles}}(\pi'^{-1}\sigma)}\mathcal{C}^k_{\sigma,\pi,\pi'}(Nx_1,y_1,Nx_2,y_2,\dots,Nx_k,y_k) \\ &\quad = N^{k-n_{\text{cycles}}(\pi'^{-1}\pi)}\mathcal{C}^k_{\sigma,\pi,\pi'}(Nx_1,y_1,Nx_2,y_2,\dots,Nx_k,y_k) . \end{aligned} \tag{5.22}$$

Thus:

$$\widetilde{\mathcal{W}}^k_{\pi,\pi'}(x_1,y_1,x_2,y_2,\dots,x_k,y_k) \sim$$
$$\sim \sum_\sigma \mathcal{C}^k_{\sigma,\pi,\pi'}(Nx_1,y_1,Nx_2,y_2,\dots,Nx_k,y_k)\prod_{i=1}^k W_1(x_i,y_{\sigma(i)})\,. \tag{5.23}$$

Then, from [9], we have:

$$0=\sum_\sigma\left[\mathcal{M}^k(N\vec{x},\vec{y},N\xi,\eta),\mathcal{C}^k_\sigma(Nx_1,y_1,\dots,Nx_k,y_k)\right]\prod_{i=1}^k W_1(x_i,y_{\sigma(i)})\,. \tag{5.24}$$

In particular, choose a permutation σ, and take the limit where $y_i\to Y(x_{\sigma^{-1}(i)})$, you get in that limit:

$$0=\left[\mathcal{M}^k(N\vec{x},Y(\vec{x_{\sigma^{-1}}}),N\xi,\eta),\mathcal{C}^k_\sigma(Nx_1,Y(x_{\sigma^{-1}(1)}),\dots,Nx_k,Y(x_{\sigma^{-1}(k)}))\right]\,. \tag{5.25}$$

Since this equation holds for any potentials V_1 and V_2, it holds for any function $Y(x)$, and thus the $Y(x_i)$'s can be chosen independentely of the x_i's, and thus, for any $y_1,\dots,y_k$, we have:

$$0=\left[\mathcal{M}^k(N\vec{x},\vec{y},N\xi,\eta),\mathcal{C}^k_\sigma(Nx_1,y_1,\dots,Nx_k,y_k)\right]\,. \tag{5.26}$$

Since it holds for any x_i's and ξ, it also holds for x_i/N and ξ/N.

Corollary 1.

$$\forall\sigma\,,\qquad [\mathcal{A}^{(k)}(\vec{x},\vec{y}),\mathcal{C}^k_\sigma(\vec{x},\vec{y})]=0\,. \tag{5.27}$$

Proof. The corollary is obtained by taking the large ξ and η limit of theorem 7 (see appendix of [9]).

5.3 Examples: $k=2$.

$$\mathcal{W}^{(2)}=\begin{pmatrix} W_{11}W_{22} & \frac{W_{11}W_{22}-W_{12}W_{21}}{(x_1-x_2)(y_1-y_2)} \\ \frac{W_{11}W_{22}-W_{12}W_{21}}{(x_1-x_2)(y_1-y_2)} & W_{12}W_{21}\end{pmatrix} \tag{5.28}$$

where $W_{ij}=W_1(x_i,y_j)$.

$$\mathcal{C}^2_{Id}=\begin{pmatrix} 1 & \frac{1}{(x_1-x_2)(y_1-y_2)} \\ \frac{1}{(x_1-x_2)(y_1-y_2)} & 0\end{pmatrix} \tag{5.29}$$

$$\mathcal{C}^2_{(12)}=\begin{pmatrix} 0 & \frac{1}{(x_1-x_2)(y_2-y_1)} \\ \frac{1}{(x_1-x_2)(y_2-y_1)} & 1\end{pmatrix}=1-\mathcal{C}^2_{Id}\,. \tag{5.30}$$

6. Application: gaussian case

There is an example of special interest, in particular for its applications to string theory in the BMN limit [26], it is the gaussian-complex matrix model case, $V_1=V_2=0$. In that case one has $E(x,y)=xy-1$, and thus:

$$W_1(x,y)=\frac{xy}{xy-1}\,. \tag{6.1}$$

The loop equation defining recursively the W_k's can be written:

$$\begin{aligned}(x_1 y_k - 1) & W_k(x_1,y_1,\dots,x_k,y_k) = \\ & = x_1 \sum_j \frac{W_{j-1}(x_j,y_1,\dots,y_{j-1}) - W_{j-1}(x_1,y_1,\dots,y_{j-1})}{x_1 - x_j} \times \\ & \quad \times W_{k-j+1}(x_j,y_j,\dots,x_k,y_k)\,. \end{aligned} \tag{6.2}$$

Its solution is then:

$$W_k(x_1,y_1,\dots,x_k,y_k) = \sum_{\sigma\in\Sigma_k} C^{(k)}_\sigma(x_1,y_1,\dots,x_k,y_k) \prod_{i=1}^k \frac{x_i y_{\sigma(i)}}{x_i y_{\sigma(i)} - 1}\,. \tag{6.3}$$

From the loop equation, one can see that $W_k(x_1,y_1,\dots,x_k,y_k)$ may have poles only when $x_i \to y_j^{-1}$ for any i and j. Because the C_σ's are rational functions of all their variables and because W_k has no singularity when $x_i = x_j$ or $y_i = y_j$, one can write:

$$W_k(x_1,y_1,\dots,x_k,y_k) = \frac{N_k(x_1,y_1,x_2,y_2,\dots,x_k,y_k)}{\prod_{i,j}(x_i y_j - 1)} \tag{6.4}$$

where $N_k(x_1,y_1,x_2,y_2,\dots,x_k,y_k)$ is a polynomial in all its variables.

Moreover, the loop equation taken for the values $x_k = 0$ or $y_k = 0$ shows that $W_k(x_1,y_1,\dots,0,y_k) = W_k(x_1,y_1,\dots,x_k,0) = 0$. Using the cyclicity property of $W_k \times (x_1,y_1,\dots,x_k,y_k)$, one can claim that it vanishes whenever one of its arguments is equal to 0. One can thus factorize the polynomial $N_k(x_1,y_1,x_2,y_2,\dots,x_k,y_k)$ as follows:

$$W_k(x_1,y_1,\dots,x_k,y_k) = \frac{Q_k(x_1,y_1,x_2,y_2,\dots,x_k,y_k)\prod_i x_i y_i}{\prod_{i,j}(x_i y_j - 1)} \tag{6.5}$$

where $Q_k(x_1,y_1,\dots,x_k,y_k)$ is a polynomial of degree $k-2$ with integer coefficient in all its variables.

Notice that $Q_k(x_1,y_1,\dots,y_{\sigma(i)}^{-1},y_i,\dots,x_k,y_k) = 0$ if σ is not planar.

As an example, we have:

- for $k=2$:

$$W_2(x_1,y_1,x_2,y_2) = \frac{x_1x_2y_1y_2}{\prod_{i,j}(x_i y_j - 1)}\,, \qquad Q_2(x_1,y_1,x_2,y_2) = 1 \tag{6.6}$$

- for $k=3$:

$$W_3(x_1,y_1,x_2,y_2,x_3,y_3) = (2 - \sum_i x_i y_{i+1} + x_1x_2x_3y_1y_2y_3)\frac{x_1x_2x_3y_1y_2y_3}{\prod_{i,j}(x_i y_j - 1)} \tag{6.7}$$

  and

$$Q_3(x_1,y_1,x_2,y_2,x_3,y_3) = (2 - x_1y_2 - x_2y_3 - x_3y_1 + x_1x_2x_3y_1y_2y_3)\,. \tag{6.8}$$

7. Conclusion

In this article, we have computed the generating functions of discs with all possible boundary conditions, i.e. the large-N limit of all correlation functions of the formal 2-matrix model. We have found that the $2k$ point correlation function can be written like the Bethe Ansatz for the δ-interacting bosons, i.e. a sum over permutations of product of 2-point functions. That formula is universal, it is independent of the potentials.

An even more powerful approach consists in gathering all possible $2k$ point correlation functions in a $k! \times k!$ matrix $\mathcal{W}^k$. We have found that this matrix $\mathcal{W}^k$ satisfies commutation relations with a family of matrices $\mathcal{M}^k$ which depend on two spectral parameters, and are related to the representations of $U(n)$ [9]. We claim that the theorem 7 is almost equivalent to the loop equations, and allows to determine $\mathcal{W}^k$.

It remains to understand how all these matrices and coefficients C_σ are related to usual formulations of integrability, i.e. how to write these in terms of Yang Baxter equations. For instance, the similarity with equations found in Razumov-Stroganov conjecture's proof [27] is to be understood.

One could also hope to find a direct proof of theorem 3, without having to solve the loop equations. In other words, we have found that the $2k$-point function can be written only in terms of W_1, while, in the derivation, we use the one point functions $Y(x)$ and $X(y)$ although they don't appear in the final result.

The next step, is to be able to compute the $1/N^2$ expansion of those correlation functions, as well as the large-N limit of connected correlation functions. We are already working that out, by mixing the approach presented in the present article and the Feynman graph approach of [14] generalized to the 2-matrix model in [15].

Another prospect is to go to the critical limit, i.e. where we describe generating functions for continuous surfaces with conformal invariance, and interpret this as boundary conformal field theory [4].

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A. Practical computation of f_σ

In this appendix, we build a set of trees in bijection with $\overline{\Sigma}_k$ and use it in order to compute practicaly the weights f_σ defined in Definition 4.

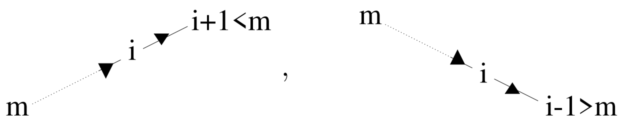
Definition 8. *Let $\mathcal{T}_k$ be the set of trees defined as follows: A tree T belongs to $T \in \mathcal{T}_k$, and is called a k-planar tree if and only if:*

- *its root is labelled $k+1$;*
- *it is composed of k+1 vertices, labeled by $[1, \dots, k+1]$;*

- *it has k edges which can be either upgoing or downgoing;*
- *its vertices have valence $1, 2$ or 3, and are of one of the following eight possibilities, in which the point m denotes the origin of the branch containing i:*
  - *two trivalent vertices:*

(A.1)

  - *four bivalent vertices:*



(A.2)

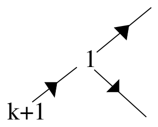
(A.3)

  - *two monovalent vertices corresponding to the leaves of the tree:*

(A.4)

Remark 3. Those trees are often called planar binary skeleton trees.

Remark 4. One can see that the first edge is necessarily upgoing, and its extremity is necessarily 1.



(A.5)

Theorem 8. *There is a bijection between $\mathcal{T}_k$ and the set of planar permutations $\overline{\Sigma}_k$:*

Proof. We build explicitly this bijection between $\overline{\Sigma}_{k-1}$ and $\mathcal{T}_{k-1}$.

Consider a planar permutation $\sigma \in \overline{\Sigma}_{k-1}$. Planarity means that σ defines a partition of the disc into faces of two kinds. Let us say that faces which correspond to cycles of $S\sigma$ are colored in white, faces which correspond to cycles of σ are colored in black.

Decompose σ and $S\sigma$ into products of irreducible cycles:

$$\sigma = \sigma_1\sigma_2\ldots\sigma_l\,, \qquad S\sigma = \tilde{\sigma}_1\tilde{\sigma}_2\ldots\tilde{\sigma}_{\tilde{l}} \tag{A.6}$$

and we assume that $\tilde{\sigma}_1$ and σ_1 contain x_1.

Because of planarity, we can define a distance of faces (i.e. cycles) to the face $\tilde{\sigma}_1$, as the number of edges one has to cross for going from a face σ_i or $\tilde{\sigma}_i$ to σ_1, and call it $D(\sigma_i)$ or $D(\tilde{\sigma}_i)$.

We also define the "origin" of a face, noted $m(\sigma_i)$ or $m(\tilde\sigma_i)$, as follows: If the face is $\tilde\sigma_1$, we define $m(\tilde\sigma_1) = k$, otherwise, because of planarity, there is only one neighbouring face which is at smaller distance of $\tilde\sigma_1$. Because of planarity, those two faces share at most one x, and the origin is defined as the label of that x.

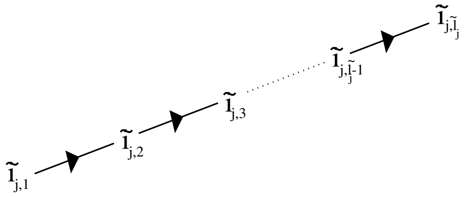
Thus, each face has a color, white or black, a distance D, and an origin m.

Now, to every face we associate a branch as follows:

- to a white face, $\tilde\sigma_j$, i.e. a cycle of $S\sigma$, noted

$$\tilde\sigma_j = (\tilde i_{j,1}, \tilde i_{j,2}, \dots, \tilde i_{j,\tilde l_j}) \,, \qquad \tilde i_{j,1} = m(\tilde\sigma_j) \,, \qquad \sigma(\tilde i_{j,n}) = \tilde i_{j,n+1} - 1 \tag{A.7}$$

  we associate the upgoing branch $\tilde i_{j,1} \to \tilde i_{j,2} \to \dots \to \tilde i_{j,\tilde l_j}$



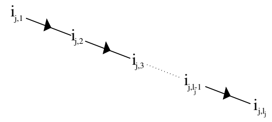
(A.8)

  (if $\tilde l_j = 1$, the sequence contains only one vertex $\tilde i_{j,1} = m(\tilde\sigma_j)$ and no edge).

- to a black face, σ_j, i.e. a cycle of σ, noted

$$\sigma_j = (i_{j,1}, i_{j,2}, \dots, i_{j,l_j}) \,, \qquad i_{j,1} = m(\sigma_j) \,, \qquad \sigma(i_{j,n}) = i_{j,n+1} \tag{A.9}$$

  we associate the downgoing branch $i_{j,1} \to i_{j,2} \to \dots \to i_{j,l_j}$



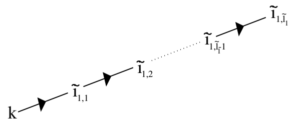
(A.10)

  (if $l_j = 1$, the sequence contains only one vertex $i_{j,1} = m(\sigma_j)$ and no edge).

- to the first face $\tilde\sigma_1$,

$$\tilde\sigma_1 = (\tilde i_{1,1}, \tilde i_{1,2}, \dots, \tilde i_{1,\tilde l_1}) \,, \qquad \tilde i_{1,1} = 1 \,, \qquad \sigma(\tilde i_{1,n}) = \tilde i_{1,n+1} - 1 \tag{A.11}$$

  we associate the upgoing branch $k \to \tilde i_{1,1} \to \tilde i_{1,2} \to \dots \to \tilde i_{1,\tilde l_1}$



(A.12)

By definition of the origin m of a face at distance D, the origin of a branch is necessarily a vertex on a branch at distance $D-1$, and from planarity, it cannot be a vertex on any other branch. Thus, there is a unique way to attach all branches to their origin, and we obtain a tree, which belongs to $\mathcal{T}_{k-1}$.

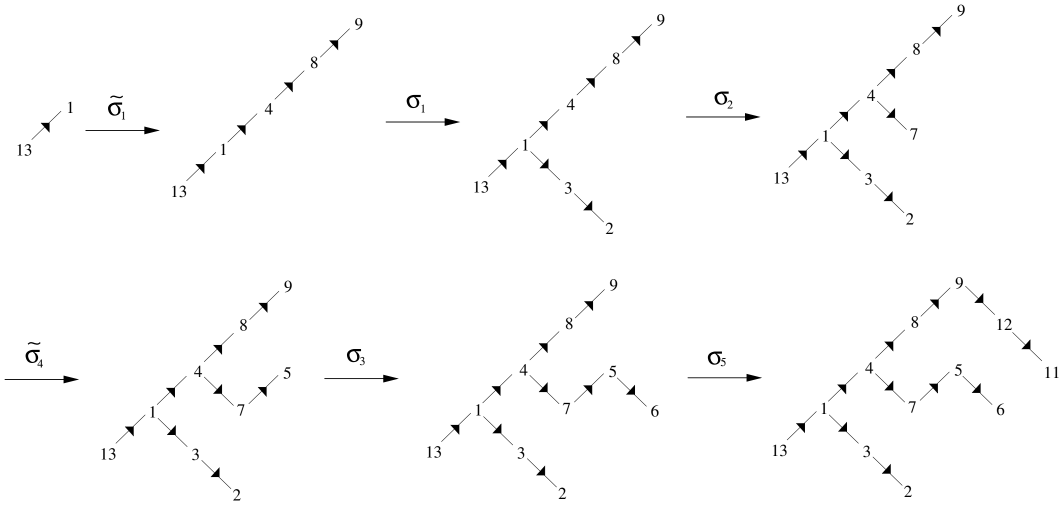
Inverse bijection: on the other hand, let us consider a $k-1$-tree. One can build a permutation $\sigma \in \overline{\Sigma}_{k-1}$ as follows: the image of an element of $(1,\ldots,k-1)$ is:

- its descendant along a downgoing propagator if it exists;
- the origin of the downgoing branch to which it belongs in the other cases.

Because of the form of the vertices, the upgoing branches are necessarily the cycles of $S\sigma$. And since the branches form a tree, it implies that two faces touch one another through zero or one edge. Thus the permutation σ is planar.

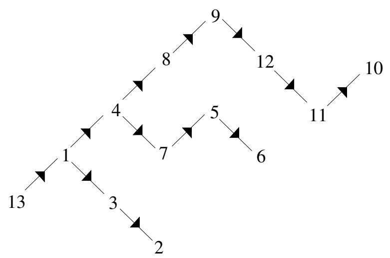
It is easy to see that this application is the inverse of the preceding one.

Example: let us carry out explicitly step by step this building for the permutation $\sigma \in \overline{\Sigma}_{12}$ introduced earlier. Notice that it is enrooted in $12+1=13$.



(A.13)

Considering the last non trivial cycle $\tilde{\sigma}_6 = (10, 11)$, one obtains finally the tree corresponding to (4.17):



(A.14)

Corollary 2. $\#\mathcal{T}_k = \mathrm{Cat}\,(k)$ *, where* $\mathrm{Cat}\,(k)$ *is the* k'^{th} *Catalan number.*

From now, one can see how to simply compute the weight f_σ associated to a permutation σ.

Consider a planar permutation $\sigma \in \overline{\Sigma}_k$ and its representation under the form of a tree $T \in \#\mathcal{T}_k$. Associate a weight to every edge of the tree as follows:

- To every downgoing edge $i_{n,j} \searrow i_{n,j+1}$ of a cycle σ_n of σ, one associates the weight $g_{i_{n,1},i_{n,j+1},i_{n,j+2}}$;
- To every upgoing edge $\tilde{i}_{n,j} \nearrow \tilde{i}_{n,j+1}$ of a cycle $\tilde{\sigma}_n$ of $S\sigma$, one associates the weight $g_{\tilde{i}_{n,j+1},\tilde{i}_{n,1},\sigma(\tilde{i}_{n,j+1})}$.

Then, the f_σ is the product of all the weights of edges composing T.

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JHEP08(2005)028